Revisiting Parameter Synthesis for One-Counter Automata

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(Joint work with Guillermo Alberto Perez [1])

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Using the control flow graph

```
x = 0
i = 0
i += x
while i >= 0:
    if i == 0:
        print("Hello world!")
    if i == 1:
        print("Lockdown = pain")
    if i == 2:
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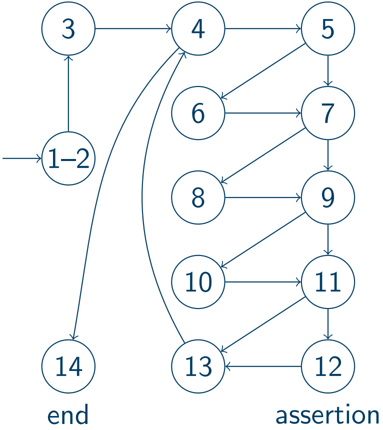

Extending the CFG with a counter

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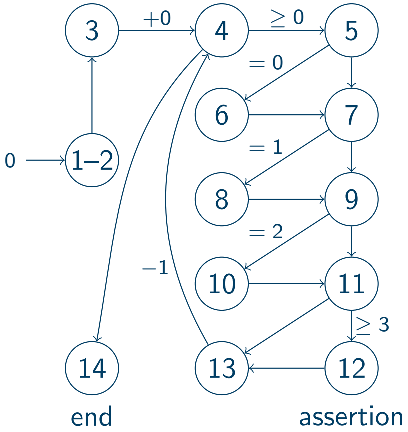

Parametric one-counter automata

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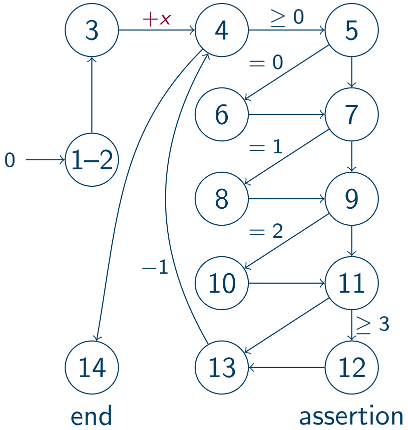


Outline

Synthesis Problems for One-Counter Automata

Presburger Arithmetic with Divisibility

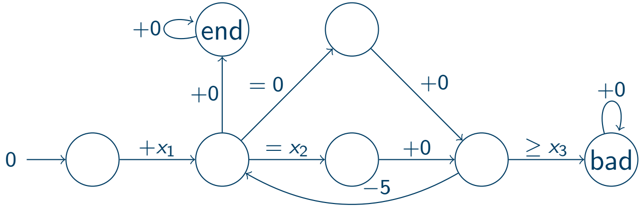
Encoding Synthesis Problems into PAD

Going back to logic: BIL

Parametric One-Counter Automata

Natural-valued parameters

$X = \{x_1, \ldots, x_n\}$



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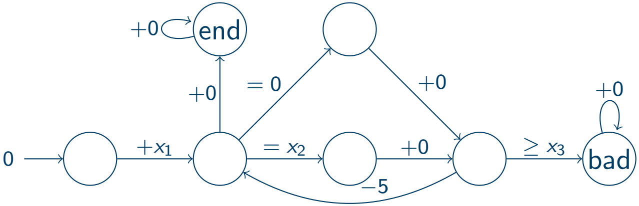

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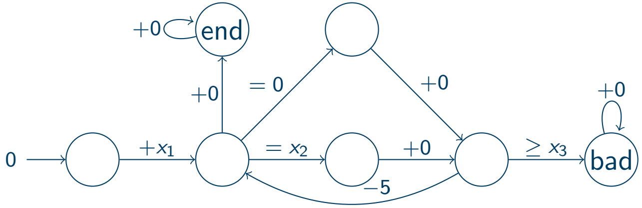

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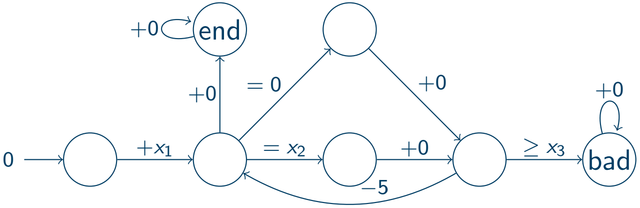

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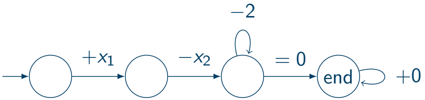
- $CU := \{+a, -a : a \in \mathbb{N}\}$, $CT := \{= a, \geq a : a \in \mathbb{N}\}$
- $PU := \{+x, -x : x \in X\}$, $PT := \{= x, \geq x : x \in X\}$

Synthesis problems

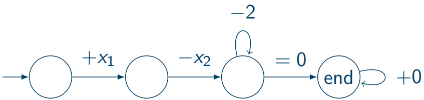
Definition (Parameter-value synthesis)

Is there some valuation $V : X \to \mathbb{N}$ such that all (infinite) runs of $\mathcal{A}$ satisfy a given ω-regular property?



	LTL	Reachability	Safety	Büchi	coBüchi
Lower bound	**PSPACE**-hard	**coNP**-hard	— $\mathbf{NP}^{\mathbf{NP}}$-hard —		
Upper bound	in **N3EXP**	— in **N2EXP** —			

Haase-Lechner approach via Logic



Logical formula: $x_1 \geq 0 \wedge x_1 \geq x_2 \wedge 2 | x_1 - x_2$

Presburger Arithmetic with divisibility!

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Full PAD is undecidable; one alternation suffices for undecidability.

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Theorem (Lipshitz'78, Lechner-Ouaknine-Worrell'15)

The existential fragment of PAD (EPAD) is decidable and in **NEXP**.

Restriction: $\forall\exists_R$PAD

- $\forall\exists_R\text{PAD} = \forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m . \varphi(\mathbf{x}, \mathbf{y})$
  - in φ, divisibilities of the form $f(\mathbf{x}) \mid g(\mathbf{x}, \mathbf{y})$

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Claim (Bozga-Iosif'05, Lechner'15)

The synthesis problems for SOCAP are decidable via an encoding into $\forall\exists_R\text{PAD}^+$.

Outline

Encoding chronology

What we need?

Formulas $\varphi(\mathbf{x})$ such that $V : X \to \mathbb{N}$ satisfies φ iff $\mathcal{A}$ has a V-run reaching t from s.

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- no positive cycles (type 1)
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Theorem (Haase et. al. '09)

If $(q, c) \rightsquigarrow (q', c')$ without zero-test, then $(q, c) \xrightarrow{\pi_1\pi_2\pi_3}$ such that, π_1, π_2 and π_3 are type 1, type 2 and type 3 reachability certificates.

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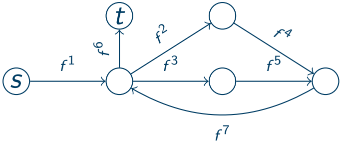

$$\bigwedge_{q \notin \{s,t\}} \left(\sum_{p \in T(\cdot,q)} f(p,q) = \sum_{r \in T(q,\cdot)} f(q,r) \right) \quad (1)$$

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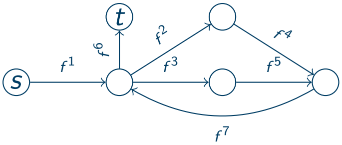

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Theorem (Euler's theorem for digraphs)

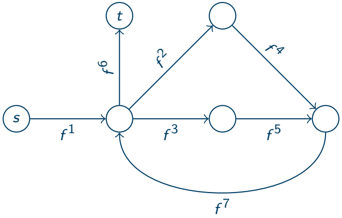
There is an s–t path iff there is a valuation of the f_i such that

- *the subgraph induced by the support and $\{(t,s)\}$ are strongly connected,*
- *it satisfies (1) and $\sum_{p \in T(\cdot,s)} f(p,s) - \sum_{r \in T(s,\cdot)} f(s,r) = 1$.*

From paths to runs

A formula per subgraph

Now $\varphi_{\text{flow}}(\mathbf{f})$ is a disjunction of the flow constraints over all subgraphs which satisfy the support condition.



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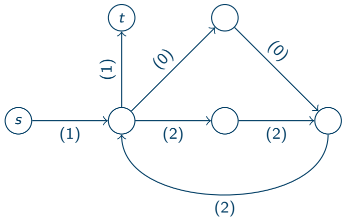

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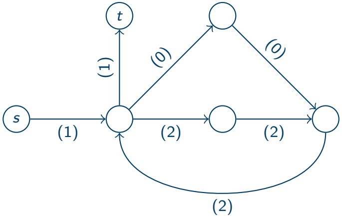

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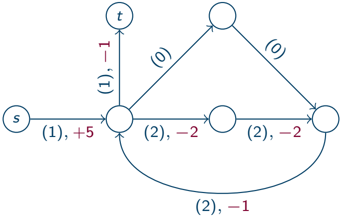

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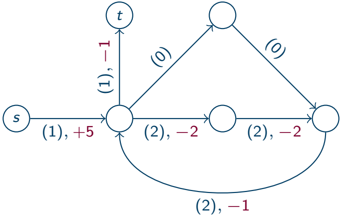
A path is not a run

Not every path can be lifted to a run because:

- of equality tests,
- the lower-bound tests,
- the counter value cannot go negative



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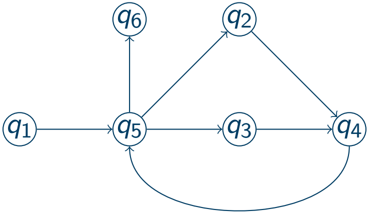

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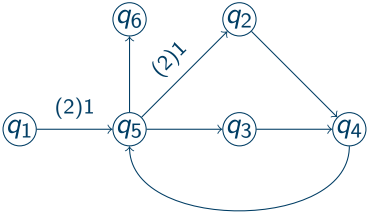

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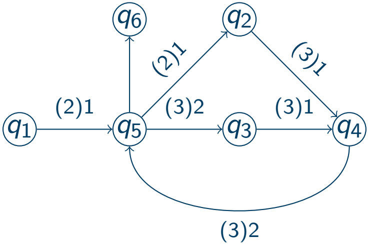

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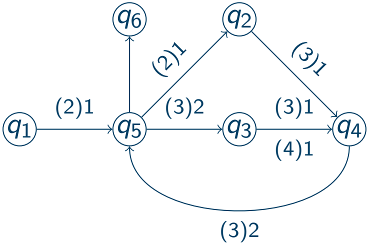

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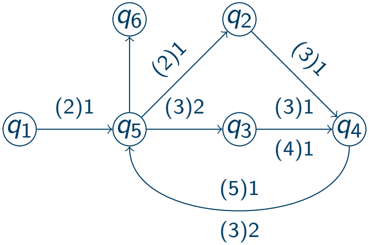

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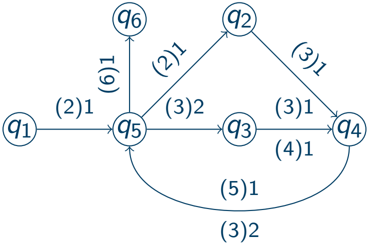

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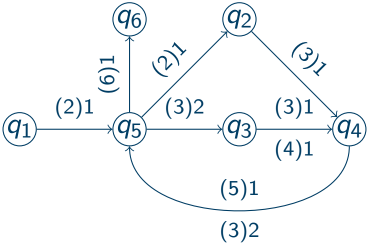

type 1: No positive cycles

To check the path can be lifted to a run:

$$\bigwedge_{m=1}^{|Q|} \sum_{i=1}^{m} \sum_{t \in T} f_i(t)\delta(t) \geq 0$$

Weight formulas using divisibility

A formula per case (1, 2, 3)

Now $\varphi_{\text{weight}}(\mathbf{x}, \mathbf{f})$ is a disjunction of the weight constraints over all decompositions.

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Divisibility to the rescue

Replace fx with a product variable z_{fx}:

$$(x_i \mid z_{f_i x_i}) \wedge (x_i > 0 \leftrightarrow z_{f_i x_i} > 0) \wedge \left(\sum z_{f_i x_i} \geq 0\right)$$

Final PAD encoding

A formula per case

For each case, we get formulas like the following:

$$\varphi(\mathbf{x}) \equiv \exists z_1 z_2 \cdots \bigvee_{\substack{\text{subgraphs}\\ \text{decomp}}} \bigwedge_{i \in I} (g_i(\mathbf{x}) \mid h_i(\mathbf{z})) \wedge \varphi_{nopos}(\mathbf{x}) \wedge \mathbf{x}, \mathbf{z} \geq \mathbf{0}$$

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The safety synthesis problem

- Positive answer if $\forall \mathbf{x}.\Phi(\mathbf{x})$ is false
- $\forall \mathbf{x}.\Phi(\mathbf{x})$ is a sentence in $\forall\exists_R \mathrm{PAD}^+$

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Claim (Bozga-Iosif'05, Lechner'15)

The synthesis problems for SOCAP are decidable via an encoding into $\forall\exists_R\text{PAD}^+$.

$\forall\exists_R$PAD & Undecidability

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Undecidable!

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A stronger restriction of $\forall\exists_R$PAD

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Theorem

The BIL fragment is decidable and in **coN2EXP**.

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Idea. Quantifier elimination (Generalized Chinese Remainder Theorem!) - Similar idea to Bozga & Iosif's work!

BIL is decidable!

Theorem (Generalized CRT)

Let $m_i \in \mathbb{N}_{>0}$, $a_i, r_i \in \mathbb{Z}$ *for* $1 \leq i \leq n$. *Then,*

$$\exists x \in \mathbb{Z}, \bigwedge_{i=1}^{n} m_i \mid (a_i x - r_i) \Leftrightarrow \bigwedge_{1 \leq i,j \leq n} \gcd(a_i m_j, a_j m_i) \mid (a_i r_j - a_j r_i) \wedge \bigwedge_{i=1}^{n} \gcd(a_i, m_i) \mid r_i$$

The solution for x *is unique modulo* $\mathrm{LCM}(m'_1, \ldots, m'_n)$, *where* $m'_i = \frac{m_i}{\gcd(a_i, m_i)}$.

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Example:

$$\forall x \exists y \bigvee_{i \in I} \bigwedge_{j \in J_i} (f_j(x) \mid (\beta_j(x) + \alpha_j(y)) \wedge f_j(x) > 0) \wedge \varphi_i(x) \wedge y \geq 0 \Rightarrow$$

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∀PAD!

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  - Idea. Reduction to Alternating 2-way automata (using idea from Bollig et.al'19)

Conclusion

	LTL	Reachability	Safety	Büchi	coBüchi
Lower bound	**PSPACE**-hard	**coNP**-hard	— $\mathbf{NP}^{\mathbf{NP}}$-hard —		
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Thank you for your attention!