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MASTERS THESIS

REACHABILITY GAMES WITH STRONG AND RELAXED ENERGY CONSTRAINTS

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Declaration

I, Ritam RAHA, declare that this thesis titled, *"Reachability Games with strong and relaxed energy constraints"* and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a masters degree(from CMI) at INRIA Rennes and CMI.
- The thesis has been prepared without resorting to plagiarism.
- Where I have consulted the published work of others, this is always clearly attributed.
- I have acknowledged all main sources of help.
- The thesis has not been submitted elsewhere for a degree.

Chennai, 01.06.2019

Ritam Raha

"The problem is
not the problem;
The problem is your attitude
about the problem."

- JACK SPARROW, *Pirates of the Caribbean*

Abstract

Energy reachability games are finite two player turn-based games played on weighted graphs. The objective of the game combines reachability objective(qualitative) with the (quantitative) requirement that the weights along a path must satisfy certain constraints (bounds). Besides having direct applications in reactive system synthesis with resource constraints, it is one of the simplest models that combine quantitative and qualitative objectives.

In this thesis, we first prove that under strict energy constraints (either only lower-bound constraint or interval constraint), those games are LOGSPACE-equivalent to energy games with the same energy constraints but without reachability objective (i.e., for infinite runs). We then consider two kinds of relaxations of the upper-bound constraints (while keeping the lower-bound constraint strict): in the first one, called *weak upper bound*, the upper bound is *absorbing*, in the sense that it allows receiving more energy when the upper bound is already reached, but the extra energy will not be stored; in the second one, we allow for *temporary violations* of the upper bound, imposing limits on the number or on the amount of violations.
We prove that when considering weak upper bound, reachability objectives require memory, but can still be solved in polynomial-time for one-player arenas; we prove that they are in coNP in the two-player setting. Allowing for bounded violations makes the problem PSPACE-complete for one-player arenas and EXPTIME-complete for two players.

Dedicated to

My Grandmothers

Mrs. Renukana Saha(Didan) & Mrs. Binapani Raha (Thamma),

for their unconditional love and support

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Contents

Chapter 1

Introduction

Games on weighted graphs. Weighted games are a common way to formally address questions related to consumption, production and storage of resources: the arena of such game are two-player turn-based games in which transitions carry positive or negative integers, representing the accumulation or consumption of resource. Various objectives have been considered for such arenas, such as optimizing the total or average amount of resources that have been collected along the play, or maintaining the total amount within given bounds. The latter kind of objectives, usually referred to as *energy objectives* [CdAHS03, BFL$^+$08], has been widely studied in the untimed setting [CD12, CDHR10, DDG$^+$10, FJLS11, JLR13, JLS15a, VCD$^+$15, BMR$^+$15, BHM$^+$17, DM18], and to a lesser extent in the timed setting [BFLM10, BLM12]. As their name indicates, energy objectives can be used to model the evolution of the available energy in an autonomous system: besides achieving its tasks, the system has to take care of recharging batteries regularly enough so as to never run out of power. Energy objectives were also used to model moulding machines: such machines inject molten plastic into a mould, using pressure obtained by storing liquid in a tank [CJL$^+$09]; the level of liquid has to be controlled in such a way that enough pressure is always available, but excessive pressure in the tank would reduce the service life of the valve.

Energy games impose strict constraints on the total amount of energy at all stages of the play. Two kinds of constraints have been mainly considered in the literature: lower-bound constraints (a.k.a. L-energy constraints) impose a strict lower bound (usually zero), but impose no upper bound; on the other hand, lower- and upper-bound constraints (a.k.a. LU-energy constraints) require that the energy level always remain within a bounded interval $[L;U]$. Finding strategies that realize L-energy objectives along infinite runs is in PTIME in the one-player setting, and in NP ∩ coNP for two players; for LU-energy objectives, it is respectively PSPACE-complete and EXPTIME-complete [BFL$^+$08]. Some works have also considered the existence of an initial energy level for which a winning strategy exist [CDHR10]. Energy objectives have also been combined with other objectives, either qualitative (e.g. parity [CD12]) or quantitative objectives (e.g. multi-dimensional energy [CDHR10, FJLS11, JLR13]).

In this paper, we focus on weighted games combining energy objectives together with reachability objectives. Our first result is the (expected) proof that L-energy games with or without reachability objectives are interreducible; the same holds for LU-energy games.

We then focus on relaxations of the energy constraints, in two different directions. In both cases, the lower bound remains unchanged, as it corresponds to running out of energy, which we always want to avoid. We thus only relax the upper-bound constraint. The first direction concerns *weak upper bounds*, already introduced in [BFL$^+$08]: in that setting, hitting the upper bound is allowed, but there is no overload, i.e. trying to exceed the upper bound will simply maintain the energy level at this maximal level. Yet, a strict lower bound is still imposed. We name these objectives LW-energy objectives. This could be used as a (simplified) model for batteries. When

considered alone, LW-energy objectives are not much different from L-energy objectives, in the sense that the aim is to find a reachable *positive loop*. LW-energy games are in PTIME for one-player games, and in NP ∩ coNP for two players. When combining LW-energy and reachability objectives, the situation changes: different loops may have different effects on the energy level, and we have to keep track of the final energy level reached when iterating those loops.
The second way of relaxing upper bounds, which we call *soft upper bound*, allows a limited number (or amount) of violations: when modeling a pressure tank, the lower-bound constraint is strict (pressure should always be available) but the upper bound is soft (excessive pressure may be temporarily allowed is needed). We consider different kinds of limits (on the number or amount of violations), and prove that (with or without reachability objectives) the energy game problems are PSPACE-complete for one-player arenas, and EXPTIME-complete for two-player ones. We also provide algorithms to optimize violations of the soft upper bound under a given strict upper bound.

Related work. Quantitative games have been the focus of numerous research articles since the 1970s, with various kinds of objectives.
Other types of quantitative games consider mean payoffs. The payoff of a run is the mean weight along this run, and the value of a mean payoff game is the maximal/minimal mean weight that one can enforce with an appropriate strategy. In this setting, one mainly has to consider the mean value of cycles, that absorb all other costs. [ZP96] shows that how the value of infinite plays can be computed.
Mean payoff games, however, only address quantitative questions as limit values of infinite runs, and do not consider quantitative constraints on prefixes of these runs, nor boolean objectives.
Discounted games are another form of quantitative games for which polynomial solutions exist. In these games, a discounting factor $\lambda < 1$ is defined, and the contribution of the i-th transition to the total payoff of a run is the weight of the transition multiplied by λ^i. The value of these games can be computed in PTIME using linear programming [And06].
Games with quantitative and reachability objectives have already been considered. [CDH17] considers total payoff, discounted payoff and energy payoff combined with a reachability objective. Given a game graph G an initial state s, and a target state t, the question addressed is whether there exists a run ρ from s to t in G such that the total payoff, discounted payoff, or the energy level of ρ is non negative. Checking that a run with payoff ≥ 0 exists for a single player, is in PTIME for all payoff functions, and for two-player games in NP ∩ coNP. For equality, the problems are respectively NP-complete and EXPSPACE-complete for total and energy payoff, and the question is still open for discounted games. The PTIME algorithms rely on the computation of values in games, which can be efficiently done as soon as one does not impose constraints on payoff achieved by prefixes of winning runs.
[CdAHS03] combines energy objectives and Büchi objectives in two players games; the objective is that a pair of components sharing resources never consume more than a certain threshold, while preserving some liveness properties, i.e. visiting some states infinitely often.
[CD12] combines quantitative and boolean objectives in Energy-Parity Games: the objective is to win a parity game while maintaining an energy constraint, i.e. ensure that the total payoff of every prefix of a winning run remains ≥ 0. Results are the following: first player 1 has winning strategies with memory of size $n \cdot d \cdot W$, where n is the number of states, W the largest weight, and d the number of priorities in the parity condition. Second, memoryless strategies are sufficient for player 2. Last memoryless winning strategies exist for player 2 if the boolean condition is a coBüchi condition.
[CDHR10] generalizes mean payoff and energy games, by considering vectors of payoff functions. These games are determined when considering finite-memory strategies. In this setting,

energy and mean payoff games are interreducible, and threshold questions for mean payoff, or unknown initial credit (whether there exist initial values for which an energy game is winning) are coNP-complete.

[CRR14] considers multiple quantitative mean payoff and energy objectives combined with parity objectives. Strategies for such quantitative games have to fulfill two objectives: first progress towards a target state or enforce a parity condition, and second, satisfy the constraint on a vector of quantitative outcomes of the runs: all mean payoff are greater than a threshold v, or all energy levels are positive. Usually these games require infinite memory. However, when restricting to finite memory strategies, the required memory is exponential.

[BFL$^+$08] considers energy games with strong lower bound constraints (L), strong lower bound and strong upper bound (LU) and strong lower bound and weak upper bound (LW) constraints. Winning runs in these games are infinite runs which prefixes satisfy the L, LU or LW energy constraint. For the one player setting, existence of strategies for L, and LW games are in PTIME, and LU games are PSPACE-complete. In the two-player setting L, and LW games are in NP $\cap$ coNP, and LU games are EXPTIME-complete.

[FJLS11] considers energy games in multi-weighted automata, and L, LW, LU games similar to those of [BFL$^+$08] but with vectors of energy levels and constraints attached to each energy level in these vectors. The complexity of these games depend on the size of energy vectors.

[HR14] considers multiple quantitative objectives, i.e. addresses the problem of determining if a player can attain a payoff in a finite union of arbitrary intervals for various payoff functions (liminf, mean-payoff, discount sum, total sum). Given a fixed number of intervals $I_1, \dots I_k$ and a payoff function f, a play π is winning if $f(\pi) \in I_j$ for some $j \in [1;k]$. The complexity of these games depend on the payoff function and ranges from NP $\cap$ coNP to EXPSPACE.

[JLS15b] considers games in multi-weighted automata and the initial credit problem, i.e. whether, starting from a vertex v, there exists a vector $B \in \mathbb{N}^k$ of initial weights such that player 1 has a winning strategy starting from configuration (v, B). They show that this problem is 2EXPTIME-complete for general multi-dimensional energy games.

[Rei16] Considers reachability in counter games under different semantics. Configurations of the game are counter values, and moves are labeled by integer vectors. The semantics of the arena either imposes no constraint on legal values of counters ($\mathbb{Z}$ semantics), forbids moves that would result in a negative value of some counter (blocking VASS semantics), or considers VASS semantics with weak lower bounds (all moves are legal, but counter values have 0 lower bound). A play is winning if it reaches a configuration from a target set. In dimension 2, under every semantics, two-player reachability games are undecidable. In dimension 1, these games are in EXPSPACE.

Chapter 2

Preliminaries

2.1 Definitions

Game arenas, Plays and Strategies. A *two-player arena* is a 3-tuple $G = \{Q_1, Q_2, E\}$ where $Q = Q_1 \uplus Q_2$ is a set of states, $E \subseteq Q \times \mathbb{Z} \times Q$ is a set of weighted edges. For $q \in Q$, we let $qE = \{(q, w, q') \in E \mid w \in \mathbb{Z},\ q' \in Q\}$, which we assume is non-empty for any $q \in Q$. A one-player arena is a two-player arena where $Q_2 = \emptyset$.

Consider a state $q_0 \in Q$. A *finite path* in an arena G from an initial state q_0 is an finite sequence of edges $\pi = (e_i)_{0 \le i < n}$ such that for every $0 \le i < n$, writing $e_i = (q_i, w_i, q'_i)$, it holds $q'_i = q_{i+1}$. Fix a path $\pi = (e_i)_{0 \le i < n}$. Using the notations above, we write $|\pi|$ for its size n of π, $\hat{\pi}_i$ for the i-th state q_i of π (with the convention that $q_n = q'_{n-1}$), and $first(\pi) = \hat{\pi}_0$ for its first state and $last(\pi) = \hat{\pi}_n$ for its last state. The empty path is a special finite path from q_0; its length is zero, and q_0 is both its first and last state. Given two finite paths $\pi = (e_i)_{0 \le i < n}$ and $\pi' = (e'_j)_{0 \le j \le n'}$ such that $last(\pi_1) = first(\pi_2)$, the concatenation $\pi_1 \cdot \pi_2$ is the finite path $(f_k)_{0 \le k < n+n'}$ such that $f_k = e_k$ if $0 \le k < n$ and $f_k = e'_{k-n}$ if $n \le k < n + n'$.

For $0 \le k \le n$, the k-th prefix of π is the finite path $\pi_{<k} = (e_i)_{0 \le i < k}$. We write $\mathsf{FPaths}(G, q_0)$ for the set of finite paths in G issued from q_0 (we may omit to mention G in this notation when it is clear from the context). Infinite paths are defined analogously; we write $\mathsf{Paths}(G, q_0)$ for the set of infinite paths from q_0.

A *strategy* for Player 1 (resp. Player 2) from q_0 is a function $\sigma\colon \mathsf{FPaths}(q_0) \to E$ associating with any finite path π with $last(\pi) \in Q_1$ (resp. $last(\pi) \in Q_2$) an edge originating from $last(\pi)$. A strategy is said *memoryless* when $\sigma(\pi) = \sigma(\pi')$ as soon as $last(\pi) = last(\pi')$.

A finite path $\pi = (e_i)_{0 \le i < n}$ *conforms* to a strategy σ of Player 1 (resp. of Player 2) from q_0 if $first(\pi) = q_0$ and for every $0 \le k < n$, either $e_k = \sigma(\pi_{<k})$, or $last(\pi_{<k}) \in Q_2$ (resp. $last(\pi_{<k}) \in Q_1$). This is extended to infinite paths in the natural way. Given a strategy σ of Player 1 (resp. of Player 2) from q_0, the outcomes of σ is the set of all paths π issued from q_0 that conform to σ.

A *game* is a triple (G, q_{init}, O) where G is a two-player arena, q_{init} is an initial state in Q, and $O \subseteq \mathsf{Paths}(G, q_{init})$ is a set of infinite paths (for Player 1), also called *objective*. A strategy for Player 1 from q_{init} is winning in (G, q_{init}, H) if its infinite outcomes all belong to O.

Payoff functions. Payoff functions are defined from the set of finite paths to integers. Many types of payoff functions have been considered in the literature. In this thesis we mainly focus on *Total-payoff* as a payoff function. We also talk about *Mean-payoff* and show how it relates to energy in our game settings.

- *Total payoff.* For a finite path $\rho = (e_i)_{0 \le i < n} \in \mathsf{FPaths}$, where $e_i = (q_i, w_i, q'_i)$ the total payoff of the path ρ is defined as, $TP(\rho) = \sum_{i=0}^{n} w_i$.

- *Finite Mean payoff.* For a finite path ρ in its usual notation, the mean payoff of the path ρ is defined as, $MP(\rho) = \frac{1}{n} \cdot TP(\rho)$.

Bounds on weights. Normally the bounds we deal with in the literature are strong bounds in the sense that they are strict and can not be violated once imposed. We use these bounds on the payoff functions of a path. Fix a payoff function, let's say total payoff. Here in this thesis, we will deal with three kinds of bounds in our game settings. :

- *Strong bounds.* As mentioned earlier, this is the usual notion of bounds used in literature. TP of a path ρ is strongly upper bounded(resp. lower) by B means $TP(\rho) \le B$(resp. $\ge B$).
- *Weak bounds.* Now, we introduce a notion of relaxing a bound calling it as weak bound. W.L.O.G, let's consider upper bound as weak. TP of a finite path $\rho = q_0 \cdot q_1, \cdots q_n$ in usual notation is weakly upper bounded by W is defined as, $TP_1 = 0, TP_{i+1} = min(TP_i + w_i, W)$. The notion of weak lower bound is analogously defined.
  So for computing $TP \upharpoonright_W (\rho)$, costs are accumulated along the transitions of ρ, but if at some point it goes above W. it is reset to W i.e. all possible increases above W are simply discarded.
- *Soft bounds.* In this thesis, we introduce another notion of relaxation of bound and name it as soft bound. Again, let's consider the soft upper bound only, the lower bound will be analogous. The notion of soft upper bound is, it can be violated but the violation is bounded.

Finite Memory & Memoryless strategies. Memory plays a very important role in Games. However, we only need to understand basic notions of finite memory strategies and memoryless ones for this thesis. A strategy for a player, $\sigma : Q^*Q_{player} \to Q$ is called a *finite-memory* strategy if every move depends on finite amount of history. The strategy is called a *memoryless* one, if it does not depend on the whole history and only depends on the current state he or she is in. Hence, a memoryless strategy can be seen as a function $\sigma : Q_{player} \to Q$.

Objectives. In this thesis, we will focus on a mixture of quantitative and qualitative objectives. The quantitative objective we name as *Energy objective*, which can be stated as follows, given single/dual strong/weak bounds, a path ρ will be winning in energy objective if starting from a designated initial vertex q_{init} with the lower bound L as the initial energy level $L + TP(\pi)$ will be always in the bound for any prefix π of ρ. On the other hand the qualitative objective here is the reachability objective that says a path is winning if it ends in the designated target vertex(or one of the vertices) q_T.

Remark 2.1.1. Taking L as the initial energy level results in no loss of generality, since any energy level can be obtained by adding a new initial vertex with an initial transition from (q_0, L).

Different Games. Now, we state what kinds of games we are going to deal with in this thesis. On the basis of bounds and objectives, we are going to analyze the following four kinds of games:

- *Energy Reachability Games with Single Bound* Given a game graph G, a starting vertex q_0 and a target vertex q_T, and a strong lower bound(w.l.o.g) $L \in \mathbb{Z}$, ER Game with single bound asks that, if starting from q_0 with L initial energy if Player 1 can reach q_T in a path ρ, such that energy level at every prefix π of ρ remains $\ge L$.

- *Energy Reachability Games with Strong Dual Bounds.* Given a game graph G, a starting vertex q_0 and a target vertex q_T, a strong lower bound L and a strong upper bound U, ER Game with strong dual bounds asks that, if starting from q_0 with L initial energy if Player 1 can reach q_T in a path ρ, such that energy level at every prefix π of ρ remains in the interval $[L, U]$.

  Energy Reachability Games with Weak Dual Bounds. Given a game graph G, a starting vertex q_0 and a target vertex q_T, a strong lower bound L and a weak upper bound W(w.l.o.g.), ER Game with weak dual bounds asks that, if starting from q_0 with L initial energy if Player 1 can reach q_T in a path ρ, such that $TP \uparrow W(\pi) \geq L$ for all prefix π of ρ.

  APNA Games.[1] Given a game graph G, a starting vertex q_0 and a target vertex q_T, a strong lower bound L, a soft upper bound S, a strong upper bound U, and a violation bound V, APNA Game asks that, if starting from q_0 with L initial energy if Player 1 can reach q_T in a path ρ, such that energy level at every prefix π of ρ remains in the interval $[L, U]$ but she can violate S at most V number of times.

2.2 Finite Mean-Payoff Reachability

In this section, lets check that what will happen if we take the mean-payoff instead of total payoff along the path. In general mean-payoff is used in theory mostly for the case of infinite games, but in this thesis we restrict ourselves to reachability, hence finite paths.

Here again w.l.o.g. we consider the upper bound as the single bound; the lower bound will be analogous. Given a game graph $G = \langle Q_1, Q_2, E, w, q_0, q_T \rangle$ and an upper bound U, the decision problem for finite mean-payoff reachability game asks, starting from q_0 with 0 initial weight, if Player 1 has a strategy to reach q_T in some path ρ such that, for all finite prefixes π of ρ, if $MP(\pi) \leq U$. Now, we state the following theorem:

Theorem 2.2.1. Finite mean-payoff reachability game with single upper bound and energy reachability game with single lower bound are inter-reducible.

Proof. Here, we will show just one side reduction, the other side will be exactly similar. We will reduce an ER game with lower bound to FMPR game with upper bound.
Let's consider a game graph $G = \langle Q_1, Q_2, E, w, q_0, T \rangle$ for energy reachability objective with lower bound 0. From this we construct $G' = \langle Q_1, Q_2, E, w', q_0, q_T \rangle$, where we change the weight function from w to w', where $w' = U - w$. We will prove that, Player 1 can win ER objective in G with lower bound 0 iff she can win FMPR objective in G' with upper bound U.

Let the winning strategy of Player 1 in G is σ. We will prove that, σ is winning in G' also. Consider an outcome ρ of σ in G' which is also an outcome of σ in G. As it is winning in G, it will surely reach q_T. Now, take any finite prefix $\pi = q_0 \cdot q_1 \cdots q_l$ of ρ. Let $|\pi| = l$ and total payoff of π

[1] I call this as "APNA Games" as promised to a certain group of friends: APNA Group. In the language Hindi, "APNA" means your own.

is $TP(\pi)$ for G and $TP'(\pi)$ for G'. As, σ is winning in G, $TP(\pi) \geq 0$. Now,

$$\begin{aligned}
MP(\pi) = &= \frac{1}{l} \cdot TP'(\pi) \\
&= \frac{1}{l} \cdot \Sigma_{i=0}^{l-1} w'(q_i, q_{i+1}) \\
&= \frac{1}{l} \cdot \Sigma_{i=0}^{l-1} (U - w(q_i, q_{i+1})) \\
&= \frac{1}{l} \cdot (l \cdot U - \Sigma_{i=0}^{l-1} w(q_i, q_{i+1}) \\
&= U - \frac{1}{l} \cdot \Sigma_{i=0}^{l-1} w(q_i, q_{i+1}) \\
&= U - \frac{1}{l} \cdot TP(\pi) \\
&\leq U
\end{aligned}$$

This shows one side of the reduction. The other side is exactly similar. □

Note that, the same reduction does not work for the dual bound case.

2.3 Thesis Description

In this thesis, we will look at the following games:
In **Chapter Three**, we will see *Energy Reachability Games with Single Bound*, in **Chapter Four**, we will see *Energy Reachability Games with Dual Bounds*, both strong and weak objectives. In **Chapter Five**, we analyze *APNA Games*.

Chapter 3

Energy Reachability Games with Single Bound

In this section, we focus on the Energy-reachability games with one strong bound. Starting from q_0 with 0 energy level, Player 1 has to reach T in a path ρ, where at all the prefixes of ρ energy level stays always over a strict lower bound or strict upper bound. W.L.O.G. we assume the strict bound is lower bound L here and also, we can take $L = 0$. We call these games as L-energy games. Note that, the infinite version of these games have been studied in [BFL+08] and has been shown to be in PTIME for 1-player case and in NP ∩ coNP in 2-player case. We first check that, how the problem changes when we add reachability instead of infinite version of the game in the next section.

3.1 L-Energy Games: Infinite vs Reachability

We prove that L-energy-reachability and L-energy infinite games are inter-reducible.

Remark 3.1.1. These results were already proven in [CDH17]; our reduction follows the same ideas as in that paper, but we develop full and direct reductions back and forth. It is worth noticing that these results are not a direct consequence of the results of [CD12] about energy parity games: in that paper, the authors focus on the *existence of an initial energy level* for which Player 1 has a winning strategy with energy-parity objectives (which encompass our energy-reachability objectives). When the answer is positive, they can compute the minimal initial energy level for which a winning strategy exists, but the (deterministic) algorithm runs in exponential time.

Reductions:

First consider a two-player arena $G = \{Q_1, Q_2, E\}$, an initial state q_{init}, and an L-energy objective. We define a new arena $G' = \{Q_1 \cup Q_c \cup \{q_T\}, Q_2, E'\}$ (assuming $q_T \notin Q$) where $Q_c = \{q_c \mid q \in Q\}$ is a copy of all the vertices of G. Note that, q_c is always a Player 1 vertex; intuitively, states in Q_c are used to allow Player 1 to stop the game and reach the target state q_T, if enough energy has been stored.
The set of transitions E' is obtained from E as follows (where the (positive) rational[1] value of ϵ will be fixed later):

- for each $(q, w, q') \in E$, there is a transition $(q, w + \epsilon, q'_c)$ and $(q'_c, 0, q')$ in E';

[1]Our definition of arenas do not allow for rational weights, but by scaling up all constants (including the energy bounds), we get an equivalent instance of our problem with only integer bounds.

- for each $q_c \in Q_c$, there is a transition $(q_c, -\delta, q_T)$ in E', where $\delta = 1 + \sum_{(q,w,q') \in E} |w|$;
- finally, E' contains an edge $(q_T, 0, q_T)$.

We claim that Player 1 has a winning strategy from q_0 for the L-energy-reachability objective in G' if, and only if, she has a winning strategy from q_0 for the L-energy objective in G.

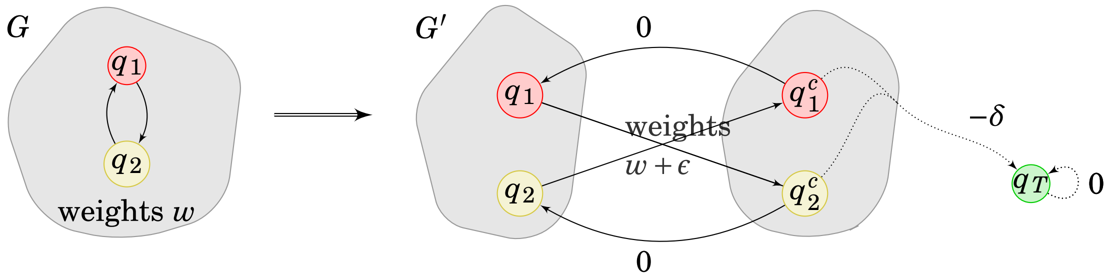


Figure 3.1: Schema of the reduction from L-energy to L-energy-reachability objectives

First assume that Player 1 has a winning strategy σ in G for the L-energy objective; then we can assume that this strategy is memoryless [BFL+08]; we define the strategy σ' as follows: for any state q of G, letting $q' = \sigma(q)$, we define $\sigma'(\pi \cdot q) = q'_c$, and

$$\sigma'(\pi \cdot q \cdot q'_c) = \begin{cases} q' & \text{if } |\pi| \leq \frac{\delta}{2\epsilon} - 1 \\ q_T & \text{otherwise.} \end{cases}$$

Obviously, any outcome μ' of σ' from q_0 reaches q_T. First note that, by construction of σ', the prefix ν' of μ' just before reaching q_T has odd length, say of length $2n-1$. Also note that it corresponds to an outcome ν of σ in G of length n. Since σ is assumed winning, ν must be L-feasible; moreover, we have

$$\tilde{\nu}'_{2i} = \tilde{\nu}'_{2i-1} = \tilde{\nu}_i + i \cdot \epsilon.$$

for all $0 \leq i < n$. Now, $\tilde{\nu}_i \geq L$ for all i, since ν is an outcome of σ, so that also $\tilde{\nu}'_i \geq L$ for all i. Moreover, $|\nu'| = \frac{\delta}{2\epsilon} - 1$ implies that, $|\nu| = \delta/\epsilon$, so that $\tilde{\nu}'_{2n-1} \geq L + \delta$, and $\tilde{\mu}'_{2n} \geq L$. It follows that σ' is winning in G' for the L-energy-reachability objective.

Conversely, assume that Player 1 wins the L-energy-reachability game G', and write σ' for a winning strategy in G' from q_0. We may assume that no negative cycle occurs along any outcome of σ': indeed, consider the (finite) execution tree of σ', and assume that it involves a negative cycle starting and ending at some state q; then there must exists a subtree rooted at q which contains no other occurrences of q; by redefining σ' so as to play as in this subtree after any occurrence of q, we remove all occurrences of our negative cycle, while preserving reachability of q_T and still satisfying the energy constraint (since removing negative cycles may only increase the energy level).
Now, take any outcome ρ' of σ' from q_0, it must eventually reach q_T. First note that, any prefix of ρ' looks like $q_0 {q_1}^c q_1 \dots q_T$. Hence, if we take any prefix π' of ρ' before reaching q_T and drop the alternate vertices, we get a corresponding path in G. Now, as ρ' eventually reaches q_T and since the edge leading to q_T has weight $-\delta$, a positive cycle must have been visited along ρ' in G'. From σ', we can then build a strategy σ that, intuitively, repeats the first positive cycle it visits(after dropping the alternate vertices). Formally, $\sigma(\pi.q) = q'$ if $\sigma'(\pi', q) = q'_c$ where π is obtained dropping alternate vertices from π' and π' contains no positive cycle. When π is a run of the form $\pi = \rho_1.\beta_1 \dots \beta_{k-1}.\rho_k$, where each β_i is a positive cycle, we take $\sigma(\rho_1.\beta_1) = \sigma(\rho_1)$. The resulting strategy σ then never takes the edge to q_T, since it only plays moves returned by σ' along outcomes that do not contain positive cycles. Moreover, all simple cycles generated

by σ in G' are positive cycles; by taking $\epsilon < \frac{1}{|Q|+1}$, these cycles still are positive cycles in G, so that σ is winning in G for the L-energy objective.

We now prove the converse reduction, which relies on similar ideas: we consider a two-player arena $G = \{Q_1, Q_2, E\}$, an initial state q_{init}, and an L-energy-reachability objective; we assume without loss of generality that there is a unique target state q_T, and write $Attr_1(q_T)$ for the Player 1-attractor of q_T in G. We build (in polynomial time) a two-player arena $G' = \{Q'_1, Q'_2, E'\}$ from G as follows:

- $Q'_1 = (Q_1 \cap Attr_1(q_T)) \cup \{q'_{init}, q_s\}$ and $Q'_2 = Q_2$. State q_{init} will serve as the new initial state, and q_s is a sink state;
- letting $E_0 = \{(q, w - \epsilon, q') \mid (q, w, q') \in E$ and $q \in Q'_1 \cup Q'_2 \setminus \{q_T\}\} \cup \{(q_T, 0, q_T), (q_s, -1, q_s)\} \cup \{(q'_{init}, q) \mid q \in \{q_{init}\} \cap Attr_1(q_T)\}$, we define $E' = E_0 \cup \{(q, 0, q_s) \mid qE_0 = \emptyset\}$. This way, all states have an outgoing edge, possibly to the sink state if no other transitions exist.

Again, the exact value of ϵ will be fixed below. We prove that Player 1 wins the L-energy-reachability game in G from q_{init} if, and only if, she wins the L-energy game in G' from q'_{init}.

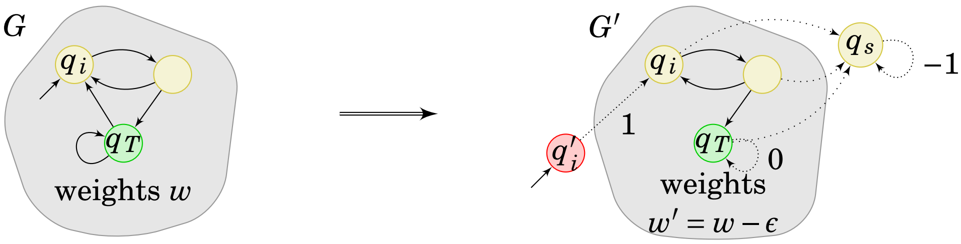


Figure 3.2: Schema of the reduction from L-energy-reachability to L-energy objectives

For the first direction, if Player 1 has a winning strategy to reach q_T from q_0 in G while maintaining the energy level above L, then she has such a strategy σ along whose outcomes the energy level is bounded above by $L + 2\delta$ (where $\delta = 1 + \sum_{(q,w,q') \in E} |w|$): indeed, if energy level $L + \delta$ is reached along some outcome, then Player 1 can achieve the reachability objective by playing her memoryless *attractor strategy*. Choosing the attractor strategy ensures reaching q_T, and will decrease the energy level by at most δ along any outcome. Similarly, following the attractor strategy can increase the energy level by no more than δ. Similarly, strategy σ can be assumed to yield no negative cycles, so that we can bound the length of the outcomes by $(\delta+1) \cdot |Q|$. Now, by taking $\epsilon < \frac{1}{(\delta+1)\cdot|Q|}$, we can mimic strategy σ in G': all outcomes only visits states in the attractor of q_T, and reach q_T in at most $(\delta + 1) \cdot |Q| + 1$ steps (the extra step is the transition from q'_{init} to q_{init}). The ϵ difference in the weights is compensated by the initial credit 1 harvested when moving from q'_{init} to q_{init}, so that all outcomes satisfy the L-energy constraint. Conversely, if Player 1 has a winning strategy σ' from q'_{init} in G', then we can assume that this strategy is memoryless [BFL+08]. Some of the outcomes may reach q_T, some may not. Since σ' is memoryless, it cannot take any negative cycle, as this would yield an outcome whose energy level tends to $-\infty$. Hence it may only take non-negative cycles in G', which correspond to positive cycles in G (since $\epsilon > 0$). As a consequence, when mimicking σ' in G, for those outcomes that do not reach q_T, the energy level will grow arbitrarily high; when it exceeds δ, Player 1 can play her attractor-strategy to reach q_T. This concludes our proof for two-player games.

Now that, we have shown that for L-energy games, infinite and reachability versions are inter-reducible, from [BFL+08] we can state the following theorem:

Theorem 3.1.2. Two-player L-energy-reachability games are decidable in NP ∩ coNP. The one-player version is in PTIME.

3.2 Conclusion

This brings us to the end of the Energy Reachability Games with Single Bounds. In this chapter we showed that, replacing infinite objective with reachability for L-energy games does not change the complexity. Hence, we have NP $\cap$ coNP complexity for this kind of games. In the next chapter, we move to the case for the similar energy games but with two bounds.

Chapter 4

Energy Reachability Games with Strong Dual Bounds

In this chapter, we will consider the quantitative games, where Player 1 has to reach his goal, always maintaining the energy level inside two bounds. This chapter we will consider both the bounds to be strong. In short, we call this **LU -Reachability Game**.

Formally, Given a game graph $G = \langle Q_1, Q_2, E, w, q_0, T\rangle$, a strong lower bound $L \in \mathbb{N}$ and a strong upper bound $U \in \mathbb{N}$, LU -reachability objective says that, with starting from q_0 with 0 initial energy, Player 1 has to reach T in a path ρ such that, for all finite prefixes ρ' of ρ, $L \leq E(\rho') \leq U$, where $E(\rho)$ is the energy level reached after taking path ρ. We will consider, both one player and two player versions of the game.

Note that, the infinite version of this game has been considered in [BFL+08] and they showed that, one player version is PSPACE-complete, where the two player version is EXPTIME-complete. In this chapter, we check what kind of changes can come if we add reachability.

4.1 One Player LU-Reachability Game

We will consider the case where $Q_2 = \phi$. Hence, all the vertices are Player 1 vertices. We will prove the following theorem:

Theorem 4.1.1. One player LU-reachability game is PSPACE-complete.

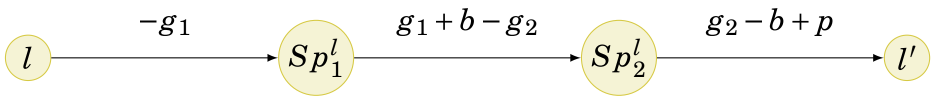
Proof. We will first show that the one player LU-reachability game is in PSPACE. Given a game graph G,the general idea is to expand the arena to G', where the vertices explicitly remember the energy levels.
Formally, given a game graph $G = \langle Q_1, Q_2, E, w, q_0, T\rangle$ and two bounds L and U, we will create a new game graph, $G' = \langle Q_1' \cup \bot, Q_2', E', q_0', T'\rangle$, where for every state $q \in Q_i$, $i \in \{1,2\}$ and for all $j \in [L,U]$, $(q,j) \in Q_i'$. Now, for all $(u,v) \in E$ with $w(u,v) = w$, we have $\langle (u,j),(v,j+w)\rangle \in E'$ for all $j \in [L,U]$ iff $j+w \in [L,U]$. Otherwise, it goes to $\bot$, a dead state. Now, Player 1 can win LU-reachability in G iff in G', $[T,j]$ for some $j \in [L,U]$ is reachable from $[q_0,0]$. Check that, as all the vertices are Player 1 vertices, we can create this new arena G' on the fly and check for a winning path. Hence, one player LU-reachability game is in PSPACE.

Now, we will prove the hardness. We will prove it by reduction from the reachability of bounded one counter automaton, which is proven to be PSPACE-complete in [FJ13].

A bounded one-counter automaton has a single counter that can store values between 0 and some bound $b \in \mathbb{N}$. The automaton may add or subtract values from the counter as long as the bounds of 0 and b are not overstepped.
For two integers $a, b \in \mathbb{Z}$ we define $[a,b] = \{n \in \mathbb{Z} : a \leq n \leq b\}$ to be the subset of integers between a and b. A bounded one-counter automaton is defined by a tuple (L, b, Δ, l_0), where L is a finite set of locations, $b \in \mathbb{N}$ is a global counter bound, specifies the set of transitions, and $l_0 \in L$ is the initial location. Each transition in Δ has the form (l, p, g_1, g_2, l'), where l and l' are locations, $p \in [-b, b]$ specifies how the counter should be modified, and $g_1, g_2 \in [0, b]$ give lower and upper guards for the transition. All numbers used in the specification of a bounded one-counter automaton are encoded in binary.
Each state of the automaton consists of a location $l \in L$ along with a counter value c. Thus, we define the set of states S to be $L \times [0, b]$. A transition exists between a state $(l, c) \in S$, and a state $(l', c') \in S$ if there is a transition $(l, p, g_1, g_2, l') \in \Delta$, where $g_1 \leq c \leq g_2$, and $c' = c + p$.

The reachability problem for bounded one-counter automata is specified as follows. An input to the problem is a pair (β, t), where β is a bounded one-counter automaton, and t is a target location. To solve the problem, we must decide whether there is a sequence of transitions between state $(l_0, 0)$ and the state $(t, 0)$.

Now, we will give reduce this problem to our game. Given the instance of a bounded one counter automaton and a target location (β, t), we construct the following graph G.
The states of the graph are exactly the locations of the counter automaton. For every transition in $(l, p, g_1, g_2, l') \in \Delta$, we create following transitions in our graph:



We add a new target t' for G and add the edge $t \xrightarrow{b} t'$. Now, in this new graph G, with upper bound b and lower bound 0 we ask, if there exists a path from q_0 to t' such that the bounds are respected.
Notice that, if in a location l, the counter value c does not follow the constraint $g_1 \leq c \leq g_2$, then here we cannot reach from l to l' as that will violate the bound constraints in Sp_1^l or Sp_2^l vertices.
Now, Player 1 wins the game in graph G iff it can reach t' maintaining the bound constraint which is possible if it reaches t with weight 0 i.e. it reaches $(t, 0)$ configuration in the one counter automata. This completes the reduction and hence the hardness result is proved. □

4.2 Two player LU-Reachability Game

Now we will move to the case of two player LU-reachability Game. Here $Q_2 \neq \phi$ anymore. We will prove the following theorem:

Theorem 4.2.1. Two players LU-reachability game is EXPTIME-complete

Proof. We first give an EXPTIME algorithm to solve two player LU -reachability game. The algorithm is exactly same as the one player case. We expand the arena to encode the energy levels in the state space instead of edges. Now, in this expanded exponential graph, the game is just reduced to check, if for some j, (T, j) is reachable from $(q_0, 0)$ or not. We know that, reachability can be solved in linear time. Hence, our game can be solved using linear time reachability algorithm on this new exponential size graph G'. Hence, we have an EXPTIME algorithm for two player LU-reachability game.

Now, we come to the hardness part. We will give a reduction from *countdown game* to our game. Countdown game has been proved to be EXPTIME-complete in [JLS07].

A countdown game C consists of a weighted graph (S,T), where S is the set of states and $T \subseteq S \times \mathbb{N} \setminus \{0\} \times S$ is the transition relation. If $t = (s,d,s') \in T$, then we say that the duration of the transition t is d. A configuration of a countdown game is a pair (s,c), where $s \in S$ is a state and $c \in \mathbb{N}$. A move of a countdown game from a configuration (s,c) is performed in the following way: first Player 1 chooses a number d, such that $0 < d \leq c$ and $(s,d,s') \in T$, for some state $s' \in S$; then Player 2 chooses a transition $(s,d,s') \in T$ of duration d. The resulting new configuration is $(s', c-d)$. There are two types of terminal configurations, i.e., configurations (s,c) in which no moves are available. If $c = 0$ then the configuration (s,c) is terminal and is a winning configuration for Player 1. If for all transitions $(s,d,s') \in T$ from the state s, we have that $d > c$, then the configuration (s,c) is terminal and it is a winning configuration for Player 2. The algorithmic problem of deciding the winner in countdown games is, given a weighted graph (S,T) and a configuration (s,c), where all the durations of transitions in C and the number c are given in binary, to determine whether Player 1 has a winning strategy from the configuration (s,c).

Given a countdown game (S,E) with initial configuration (s_0,c_0), we construct a weighted game as follows: let $S_1 = S$, $S_2 = \{(s,d) | (s,d,s') \in E\}$, T a target state and

$$T = \{s \xrightarrow{d} (s,d) | (s,d) \in S_2\} \cup \{(s,d) \xrightarrow{0} s' | (s,d,s') \in E\} \cup \{s \xrightarrow{-c_0} T | s \in S\}$$

The upper bound is set to c_0 and the lower bound is 0 . Player 1 can now from a state $s \in S$ choose a particular number d and Player 2 from the temporary state (s,d) choose a transition to a state $s' \in S$. The number d is added to the accumulated weight and the same repeats. As the accumulated weight is bounded by c_0 , Player 1 has to eventually take some transition labeled $-c_0$ and reach the target state T . In order not to drop below zero, this is only possible if the accumulated weight is exactly c_0 , hence the first player in the countdown game has a winning strategy if and only if Player 1 has a winning strategy in the two player LU-reachability game. □

4.3 Conclusion

In this chapter we have seen that for LU-reachability game, one player version is PSPACE-complete and two player version is EXPTIME-complete. Hence, we can state the following theorem from the observation of Chapter 3 and Chapter 4:

Theorem 4.3.1. If the bounds are strong, both with single and dual bound infinite and reachability version of energy games are inter-reducible.

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From the next chapter on wards, we will move to the cases, when one of the bound is weak(relaxed) and the other bound remains strict.

Chapter 5

Energy Reachability Games with Weak Dual Bounds

In this chapter, we will explore the dual bound quantitative reachability games(LW games) where one bound is weak. Recall the notion of weak bound: a bound(w.l.o.g say, upper bound) U is weak means, if the weight hits U, it never goes above U, it stays at U until it goes lower. Also recall that, the weight of a path γ with weak lower bound U is denoted as $w{\uparrow}_U(\gamma)$. Here we will consider the case where the lower bound is strong and the upper bound is weak. Note that, the other case can be obtained just by reversing the sign of the weights. Infinite version of LW-games have been studied in [BFL+08]. The objective for player 1 there is to find an infinite path, where strong lower bound and weak upper bound constraints are maintained. This problem is conceptually easy: they show that it is in PTIME for one-player arenas, and in NP ∪ coNP for two-player arenas in the same paper. In this chapter, at first we see why adding reachability makes the LW-energy game harder and then later we try to solve these games.

5.1 LW-energy Games: Infinite vs Reachability

LW-energy infinite games require player 1 to find an infinite path which is feasible under LW constraints; it suffices to find a cycle that can be iterated once with a positive effect. It follows that memoryless strategies are enough for both the players.
The situation is different when we have a reachability condition: players may have to keep track of the exact energy level in order to find their way to the target state. Lets see an example:

Example 5.1.1. Consider the one-player arena of Fig. 5.1, where the lower bound is $L = 0$ and the weak-upper bound is $W = 5$, and the target state is q_T. Starting with initial credit 0, we first have to move to q_1, and then iterate the positive cycle $\beta_1 = (q_1, 2, q_2).(q_2, -2, q3).(q_3, +1, q_1)$ of total weight +1 three times, ending up in q_1 with energy level 3. We then take the cycle $\beta_2 = (q_1, +2, q_2).(q_2, -5, q4).(q_4, +5, q_1)$, which raises the energy level to 5 when we come back to q_1, so that we can reach q_T. Notice that β_1 has to be chosen 3 times before taking cycle β_2, and that repeating β_1 more than 4 times maintains the energy level at 4, which is not sufficient to reach q_T. This shows that one cannot rely on a single cycle to win a LW-energy reachability game.

Now, we try to solve LW-energy reachability games. Like the strong bound games, here also we first solve the one player version of the game and then move to the two player's case.

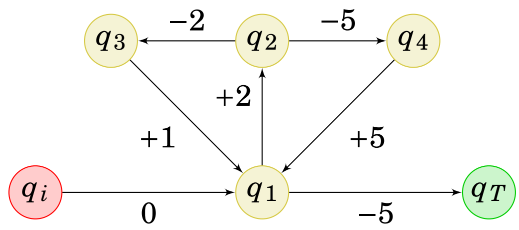

Figure 5.1: A one-player arena with LWenergy reachability objective

5.2 One Player QR Games with Weak Dual Bounds

We consider the one player version of this game, where $Q_2 = \phi$. We will prove the following theorem:

Theorem 5.2.1. Given a game graph G , a weak upper bound W and a strong lower weak bound 0, deciding if P_1 can win the one player QR games with weak dual bound game in G is in PTIME.

Proof. Before proving it formally, let us look at some intuition: consider a winning strategy σ of P_1. Intuitively, any outcome of σ will not have any negative or zero cycle, as player 1 can just ignore the cycle and still win. Hence, it will be either an acyclic path maintaining the objective, or she has to choose a negative cycle where she can rotate enough number of times maintaining the objective, lowers the energy to a certain stable value and then continues forward along the path.

Now, we examine a positive cycle in a graph from a vertex v. Starting, from some initial energy e_{init} from v, we will reach v_1 in the cycle, where the energy is the lowest, say e_{min}. Then, the energy level decreases along the cycle and reaches v_2, where the energy level is highest in the cycle,let's say M. Then, it goes back to v, with energy, say e_{out}. Let $m = M - e_{out}$. Now, if player 1 can reach vertex v, with at least $a = e_{init} - e_{min}$ energy, she will be able to rotate through this negative cycle many times as it will never violate the strong lower bound constraint. Now, as the weak upper bound is W, after sufficient number of rotation, she can reach v_2 with energy level W and reach v with $W - m$ amount of energy. The phenomena has been depicted with an example in Figure 5.2.

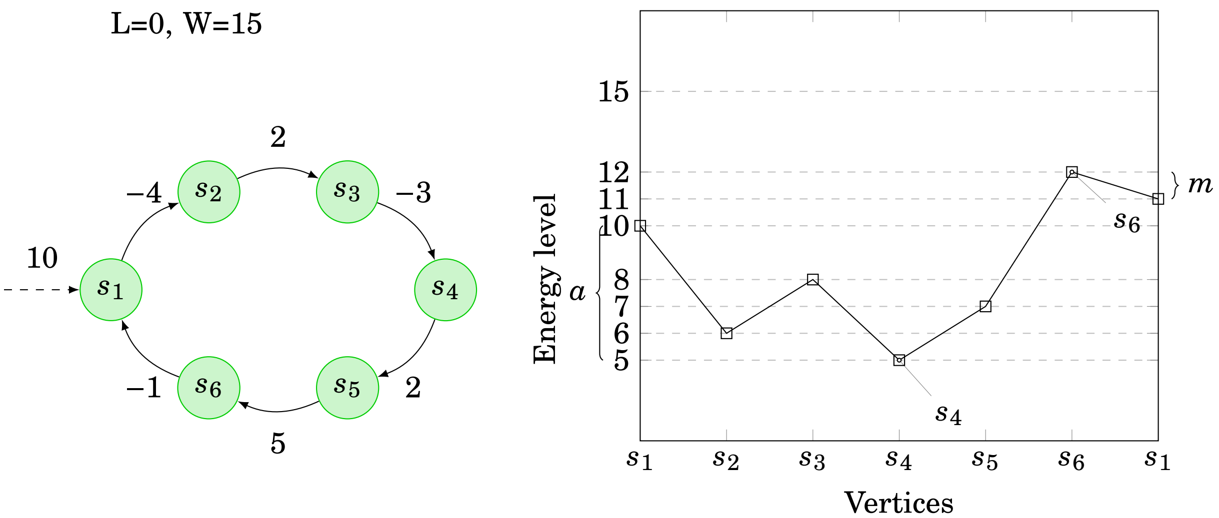

Figure 5.2: Energy level of a positive cycle

In the example, $a = 5$ and $m = 1$, i.e. if player 1 can reach s_1 with at least 5 energy level, she can rotate the cycle as much as she wants, and can increase the output energy level up to 14.

Now, we express this intuition formally by stating a series of lemmas; we have given a proof-sketch of all the lemmas in this chapter, all the detailed proofs have been written in the Appendix portion of the thesis.

Lemma 5.2.2. Let π be a finite path in a one-player arena G. If $(q,u) \xrightarrow{\pi}_{LW} (q',u')$, then for any $v \geq u$, $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ for some $v' \geq u'$.

Notice that, even if we add condition $u' > u$ in the hypotheses of Lemma 5.2.2, it need not be the case that $v' > v$. In other terms, a sequence of transitions may have a positive effect on the final energy level from some configuration, and a negative effect from another one, due to the weak upper bound limit. Below, we prove a series of results related to this issue, and that will be useful for the rest of the proof.

Lemma 5.2.3. Let π be a finite path in a one-player arena G, and consider two LW-runs $(q,u) \xrightarrow{\pi}_{LW} (q',u')$ and $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ with $u \leq v$. Then $u' - u \geq v' - v$, and if the inequality is strict, then the energy level along the run $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ must have hit W.

Lemma 5.2.4. Let π be a finite path in a one-player arena G, for which there is an LW-runs $(q,u) \xrightarrow{\pi}_{LW} (q',u')$. If u' is the maximal energy level along that run, then $(q,W) \xrightarrow{\pi}_{LW} (q',W)$; if u is the maximal energy level along the run above, then $(q,W) \xrightarrow{\pi}_{LW} (q',W+u'-u)$.

Lemma 5.2.5. Let π be a finite path in a one-player arena G. If $(q,u) \xrightarrow{\pi}_{LW} (q',u')$ with $u' > u$ and $(q,w) \xrightarrow{\pi}_{LW} (q',w')$ with $w' > w$, then for any $u \leq v \leq w$, it holds $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ with $v' > v$.

From Lemma 5.2.2, it follows that any run witnessing LW-energy reachability can be assumed to contain no cycle with non-positive effect. Formally:

Lemma 5.2.6. Let π be a finite path in a one-player arena G. If $(q,u) \xrightarrow{\pi}_{LW} (q',u')$ and π can be decomposed as $\pi_1 \cdot \pi_2 \cdot \pi_3$ in such a way that $(q,u) \xrightarrow{\pi_1}_{LW} (s,v) \xrightarrow{\pi_2}_{LW} (s,v') \xrightarrow{\pi_3}_{LW} (q',u')$ with $v' \leq v$, then $(q,u) \xrightarrow{\pi_1 \cdot \pi_3}_{LW} (q',u'')$ with $u'' \geq u'$.

The following lemma states that any LW-feasible cycle with non-negative effect can be iterated, and that the energy level reached after a certain number of iterations eventually converges.

Lemma 5.2.7. Let π be a cycle on q such that $(q,u) \xrightarrow{\pi}_{LW} (q,v)$ for some $u \leq v$. Then $(q,u) \xrightarrow{\pi^{W-L}}_{LW} (q,v')$ for some v', and $(q,v') \xrightarrow{\pi}_{LW} (q,v')$.

Fix a path π in G, and assume that some cycle ϕ appears (at least) twice along π: the first time from some configuration (q,u) to some configuration (q,u'), and the second time from (q,v) to (q,v'). First, we may assume that ϕ has length at most $|Q|$, since otherwise we can take an inner sub-cycle. We may also assume that $v > u'$, as otherwise we can apply Lemma 5.2.6 to get rid of the resulting non-positive cycle between (q,u') and (q,v). For the same reason we may assume $u' > u$ and $v' > v$. As a consequence, by Lemma 5.2.5, by repeatedly iterating ϕ from (q,u), we eventually reach some configuration (q,w) with $w \geq v'$, from which we can follow the suffix of π after the second occurrence of ϕ. It follows that all occurrences of ϕ along π can be grouped together, and we can restrict our attention to runs of the form $\alpha_1 \cdot \phi_1^{n_1} \cdot \alpha_2 \cdot \phi_2^{n_2} \cdots \phi_k^{n_k} \cdot \alpha_{k+1}$ where the cycles ϕ_j are distinct and have size at most $|Q|$, and the finite runs α_j are acyclic. Notice that by Lemma 5.2.7, we may assume $n_j = W - L$ for all j.
While this allows us to only consider paths of a special form, this does not provide *short* witnesses, since there may be exponentially many cycles of length less than or equal to $|Q|$, and the

witnessing run may need to iterate several cycles looping on the same state (see Example 5.1.1). In order to circumvent this problem, we have to show that all cycles need not be considered, and that one can compute the "useful" cycles efficiently. For this, we first introduce *universal* cycles, which are cycles that can be iterated from any initial energy level greater than the lower bound L.

Definition 5.2.8. A *universal cycle* on q is a cycle ϕ with $first(\phi) = last(\phi) = q$ such that $(q, L) \xrightarrow{\phi}_{LW} (q, v_{\phi,L})$ for some $v_{\phi,L}$. A universal cycle is *positive* if $v_{\phi,L} > L$

Note that, when a cycle ϕ is iterated $W - L$ times in a row, then some universal cycle σ is also iterated $W - L - 1$ times (by considering the state with minimal energy level along ϕ). In figure 5.3, $abcde$ is not a universal cycle as it can not be iterated from initial energy level 0, but $bcdea$ is a universal one as it can be iterated infinitely from initial energy level 0. Also, note that b is actually the lowest point of the cycle.

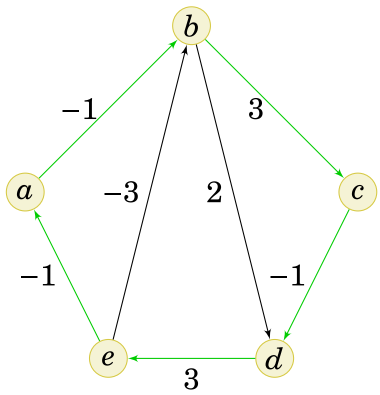

Figure 5.3: Universal Cycles: abcde is not universal but bcdea is

Hence, every cycle can be replaced by a universal cycle. As a consequence, iterating only universal cycles is enough: we may now only look for runs of the form $\beta_1 \cdot \sigma_1^{n_1} \cdot \beta_2 \cdot \sigma_2^{n_2} \cdots \sigma_k^{n_k} \cdot \beta_{k+1}$ where σ_j's are *universal* cycles of length at most $|Q|$. Now, assume that some state q admits two universal cycles σ and σ', and that both cycles appear along a given run π. Write e (resp. e') for the energy levels reached after iterating σ (reap. σ') $W - L$ times. We define an order on universal cycles of q by letting $\sigma \rhd \sigma'$ when $e > e'$. Then if $\sigma \rhd \sigma'$, each occurrence of σ' along π can be replaced with σ, yielding a run π' that still satisfies the LW-energy condition (and has the same first and last states). in figure 5.3, both $bcdea$ and bde are universal cycles on the vertex b. But, for $bcdea$, output energy is 2 while for bde it is 1. Hence, $bcdea \rhd bde$.
Generalizing this argument, each state that admits universal cycles has an optimal universal cycle of length at most $|Q|$, and it is enough to iterate only this universal cycle to find a path witnessing reachability. This provides us with a *small witness*, of the form $\gamma_1 \cdot \tau_1^{W-L} \cdot \gamma_2 \cdot \tau_2^{W-L} \cdots \tau_k^{W-L} \cdot \gamma_{k+1}$ where τ_j are optimal universal cycles of length at most $|Q|$ and γ_j are acyclic paths. Since it suffices to consider at most one universal cycle per state, we have $k \leq |Q|$. From this, we immediately derive an NP algorithm for solving LW-energy reachability for one-player arenas: it suffices to non-deterministically select each portion of the path, and compute that each portion is LW-feasible (notice that there is no need for checking universality nor optimality of cycles; those properties were only used to prove that small witnesses exist). Checking LW-feasibility requires computing the final energy level reached after iterating a cycle $W - L$ times; this can be performed by detecting the highest energy level along that cycle, and computing how much the energy level decreases from that point on until the end of the cycle.

We now prove that optimal universal cycles of length at most $|Q|$ can be computed for a given state q_0. For this we unwind the graph from q as a DAG of depth $|Q|$, so that it includes all cycles of length at most $|Q|$. We name the states of this DAG $[q',d]$ (using square brackets to avoid confusion with configurations (q,l) where l is the energy level) where q' is the name of a state of the arena and d is the depth of this state; hence there are transitions $([q',d],w,[q'',d+1])$ in the DAG as soon as there is a transition (q',w,q'') in the arena.
We then explore this DAG from its initial state $[q_0,0]$, looking for (paths corresponding to) universal cycles. Our aim is to keep track of all runs from $[q_0,0]$ to $[q',d]$ that are prefixes of universal cycles starting from q_0. Actually, we do not need to keep track of those runs explicitly, and it suffices for each such run to remember the following two values:

- the maximal energy level M that has been observed along the run so far (starting from energy level L, with weak upper bound W);
- the difference m between the maximal energy level M and the final energy level in $[q',d]$. Notice that $m \geq 0$, and that the final energy level in $[q',d]$ is $M-m$.

Figure 5.4 shows two example cycles. The first one ends with $M_1 = 5, m_1 = 4$, i.e. with an energy level of 1. The second cycle has a maximal energy level $M_2 = 4$ and ends with $m_2 = 2$. Hence, iterating this cycle, one can end in state q_0 with energy level $W - m_2 = 3$.

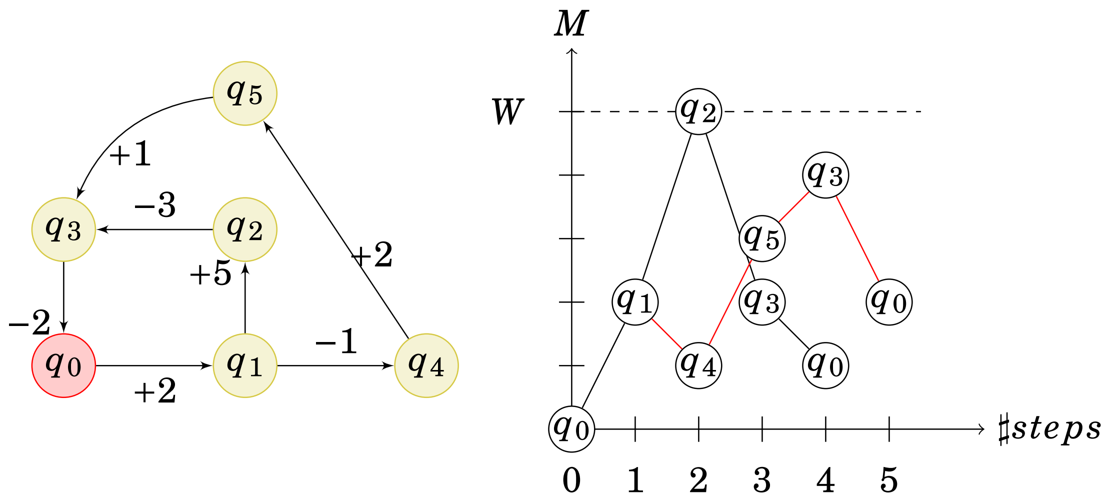

Figure 5.4: Two cycles with upper bound $W = 5$

If we know the values (M,m) of some path from $[q_0,0]$) to $[q',d]$, we can decide if a given transition with weight w from $[q',d]$ to $[q'',d+1]$ can be taken (the resulting path can still be a prefix of a universal cycle if $M-m+w \geq L$), and how the values of M and m have to be updated: if $w > m$, the run will reach a new maximal energy level, and the new pair of values is $(min(W;M-m+w),0)$; if $m+L-M \leq w \leq m$, then the transition can be taken: the new energy level $M-m+w$ will remain between L and M, and we update the pair of values to $(M,m-w)$; finally, if $w < m+L-M$, the energy level would go below L, and the resulting run would not be a prefix of a universal cycle.
Following these ideas, we inductively attach to states of the DAG sets of labels: initially, $[q_0,0]$ is labelled with $(M = L, m = 0)$; then if a state $[q',d]$ is labelled with (M,m), and if there is a transition from $[q',d]$ to $[q'',d+1]$ with weight w:

- if $w > m$, then we label $[q'',d+1]$ with the pair $(\max(W;M-m+w),0)$;
- if $m+L-M \leq w \leq m$, we label $[q'',d+1]$ with $(M,m-w)$.

Lemma 5.2.9. Let $[q,d]$ be a state of the DAG, and M and m be two integers such that $0 \le m \le M$. Upon termination of this algorithm, state $[q,d]$ of the DAG is labelled with (M,m) if, and only if, there is an LW-run of length d from (q_0,L) to $(q,M-m)$ along which the energy level always remains in the interval $[L,M]$.

Lemma 5.2.10. Let $[q_0,d]$ be a state of the DAG, with $d > 0$. Let m be a non-negative integer such that $L < W - m$. Upon termination of this algorithm, state $[q_0,d]$ is labelled with (M,m) such that $M - m > L$ if, and only if, there is a universal positive cycle ϕ on q_0 of length d such that $(q_0,L) \xrightarrow{\phi^{W-L}}_{LW} (q_0,W-m)$.

The algorithm above computes optimal universal cycles, but it still runs in exponential time (in the worst case) since it may generate exponentially many different labels in each state $[q,d]$ (one per path from $[q_0,0]$ to $[q,d]$). We now explain how to only generate polynomially-many pairs (M,m). This is based on the following partial order on labels: we let $(M,m) \preceq (M',m')$ whenever $M - m \le M' - m'$ and $m' \le m$. Notice in particular that

- if $M = M'$, then $(M,m) \preceq (M',m')$ if, and only if, $m' \le m$;
- if $m = m'$, then $(M,m) \preceq (M',m')$ if, and only if, $M \le M'$.

We then have the following lemma:

Lemma 5.2.11. Consider two paths π and π' such that $first(\pi) = first(\pi')$ and $last(\pi) = last(\pi')$, and with respective values (M,m) and (M',m') such that $(M,m) \preceq (M',m')$. If π is a prefix of a universal cycle ϕ, then π' is a prefix of a universal cycle ϕ' with $\phi' \rhd \phi$.

In our algorithm above, it suffices to keep track of the maximal labels for $\preceq$, since our aim is to compute optimal universal cycles. It remains to prove that this way, we only store a polynomial number of labels:

Lemma 5.2.12. If the DAG construction only stores maximal labels (for $\preceq$), then it runs in polynomial time.

Using the algorithm above, we can compute, for each state q of the original arena, the smallest value m_q for which there exists a universal cycle on q that, when iterated sufficiently many times, leads to configuration $(q, W - m_q)$. Since universal cycles can be iterated from any energy level, if q is reachable, then it is reachable with energy level $W - m_q$. We make this explicit by adding to our arena a special self-loop on q, labelled with $set(W - m_q)$, which sets the energy level to $W - m_q$ (in the same way as *recharge transitions* of [EF13]).
In the resulting arena, we know that we can restrict to paths of the form $\gamma_1 \cdot \nu_1 \cdot \gamma_2 \cdot \nu_2 \cdots \nu_k \cdot \gamma_{k+1}$, where ν_i are newly added transitions labelled with $set(W-m)$, and γ_i are acyclic paths. Such paths have length at most $(|Q|+1)^2$. We can then inductively compute the maximal energy level that can be reached (under our LW-energy constraint) in any state after paths of length less than or equal to $(|Q|+1)^2$. This can be performed by unwinding (as a DAG) the modified arena from the source state q_{init} up to depth $(|Q|+1)^2$, and labelling the states of this DAG by the maximal energy level with which that state can be reached from (q_{init},L); this is achieved in a way similar to our algorithm for computing the effect of universal cycles, but this time only keeping the maximal energy level that can be reached (under LW-energy constraint). As there are at most $|Q|$ states per level in this DAG of depth at most $(|Q|+1)^2$.
This completes the proof for our theorem. □

Now, let's look at the following example:

Example 5.2.13. Consider the one-player arena of Fig. 5.5. We assume $L = 0$, and fix an even weak upper bound W. The state s has $W/2$ disjoint cycles: for each odd integer i in $[0;W-1]$, the cycle c_i is made of three consecutive edges with weights $-i$, $+W$ and $-W+i+1$. Similarly, the state s' has $W/2$ disjoint cycles: for even integers i in $[0;W-1]$, the cycle c'_i has weights $-i$, $+W$ and $-W+i+1$. Finally, there are: two sequences of k edges of weight 0 from s to s' and from s' to s; an edge from the initial state to s with weight 1; an edge from s' to the target state with weight $-W$. The total number of states then is $2W+2k+2$.
In order to go from the initial state, with energy level 0 to the final state, we have to first take the cycle c_1 (with weights -1, $+W$, $-W+2$) on s (no other cycles c_i can be taken). We then reach configuration $(s,2)$. Iterating c_1 would have no effect, and the only next interesting cycle is c_2, for which we have to go to s'. After running c_2< we end up in $(s',3)$. Again, iterating c_2 has no effect, and we go back to s, take c_3, and so on. We have to take each cycle c_i (at least) once, and take the sequences of k edges between s and s' $W/2$ times each. In the end, we have a run of length $3W+Wk+2$.

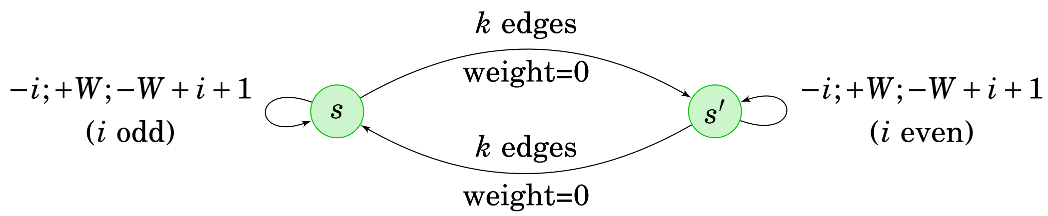

Figure 5.5: An example showing that more than one cycle per state can be needed.

Let us look at the universal cycles that we have in this arena: besides the cycles made of the $2k$ edges with weight zero between s and s', the only possible universal cycles can only depart from the first state of each cycle c_i (as there are the only states having a positive outgoing edge).
As we proved, such cycles can be iterated arbitrarily many times, and set the energy level to some value in $[L;w]$. Since the only edge available at the end of a universal cycle has weight $+W$, the exact value of the universal cycles is unimportant: the energy level will be W anyway when reaching the second state of each cycle c_i. As a consequence, using set-edges in this example does not shorten the witnessing run, which then cannot be shorter than $3W+Wk+2$ (which is more than $2|Q|$ but less than $|Q|^2$). This demonstrates that we cannot avoid looking for quadratic-size runs in the modified arena at the end of our algorithm.

Now, we will move to the case for the two player version.

5.3 Two Player QR Games with Weak Dual Bounds

Now, $Q_2 \neq \phi$ anymore. We begin with proving a result similar to Lemma 5.2.2:

Lemma 5.3.1. Let G be a two-player arena, equipped with an LW-energy-reachability objective. Let q be a state of G, and $u \le u'$ in $[L;W]$. If Player 1 wins the game from (q,u), then she also wins from (q,u').

By Martin's theorem [Mar75], our games are determined. It follows that if Player 2 wins from some configuration (q,v), she also wins from (q,v') for all $L \le v' \le v$ (assuming the contrary, i.e. (q,v') winning for Player 1, would lead to the contradictory statement that (q,v) is both winning for Player 1 and Player 2). We now prove that Player 1 may need exponential memory, while Player 2 can play memoryless strategies:

Lemma 5.3.2. For two player QR games with weak dual bounds, exponential memory may be necessary for player 1. For player 2, memoryless strategies are sufficient.

Proof. As reachability is one of the objective, trivially finite memories are sufficient for both the players. Now, we will show, a class of game graphs, where exponential memory may be necessary for player 1.

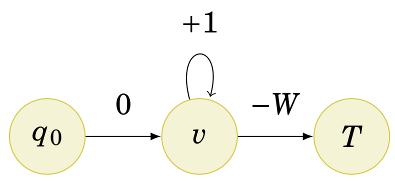

Figure 5.6: Exponential memory is necessary for player 1

In the game graph of Figure 5.6, all the vertices are player 1 vertices, strong lower bound is 0 while weak upper bound is W. It is easy to see, player 1 needs at least W- memory to win this game. Considering binary encoding. the value of W is exponential in the size of the input.

Now, We now prove that Player 2 has memoryless optimal strategies. According to Lemma 5.3.1, for each state q, there is an integer $v_q \in [L;W+1]$ such that Player 1 wins the game from any configuration (q,v) satisfying $v_q \leq v \leq W$, while Player 2 wins the game from any configuration (q,v) with $L \leq v < v_q$.
Assume that Player 2 wins the game from some state (q,v), with $L \leq v \leq v_q$. Denote with $(q,p_i,q_i)_{1\leq i\leq m}$ for the set of outgoing transitions from q. By definition of v_{q_i}, Player 1 wins the game from any configuration of the form (q_i,v) with $v \geq v_{q_i}$. Since Player 2 wins from (q,v), there must exist an index $1 \leq i \leq m$ such that $v + p_i \leq v_{q_i}$. This defines a winning move for Player 2 from (q,v). The same argument applies in all states, and yields a memoryless winning strategy for Player 2. □

Now, based on the previous lemma we can conclude the following theorem:

Theorem 5.3.3. Given a game graph G , a strong upper bound U and a lower weak bound 0, deciding if P_1 can win the 2 player QR games with weak dual bound game in G is in $coNP$.

Proof. A priori, lemma 5.3.2 says if P_2 has a winning strategy, he has a memoryless one. Now, we can guess a strategy for Player 2 (i.e. select one transition per state in Q_2) and verify that this strategy is winning for Player 2 by solving the one player game, which has been proved to be in polynomial time in theorem 5.2.1. □

5.4 Conclusion

In this chapter, we have showed that QR game with weak dual bounds is in PTIME for the single player case and is in coNP for the two player case. We also believe that the two player version of the game is in NP and not coNP-hard.
This ends all the QR game versions with bounds on weight functions. In the next chapter, we will introduce a new QR game, with a notion of bound on number of violations.

Chapter 6

Apna Game

In the previous chapters, we have explored energy games, where the energy levels were bounded. In Chapter 5, we have seen one kind of relaxation of bounds. In this chapter, we will explore a new kind of a game. Here also, we have lower and upper bounds on weights and one bound (W.L.O.G. lower bound) is strong, but the idea of relaxation of the other bound is different. In this game, there is a soft upper bound and a strong upper bound. Player 1 is allowed to violate the soft upper bound, but the number of violations is bounded. But, with bounded violations also, she is not allowed to violate the strong lower bound L and strong upper bound U. The number of vertices, he traverses along his path with weights higher than the upper bound can be taken as one of the simplest **violation measures**. In this chapter, we have defined in total three kinds of violation measure. We call this bounded violations reachability game as *Apna Game*. Let's formally define the game in the following section.

6.1 Description of the Game

Consider a game graph $G = \langle Q_1, Q_2, E, w, q_0, T\rangle$, two strict bounds L and $U \in \mathbb{Z}$, a soft upper bound S, a threshold $V \in \mathbb{Z}$. For an LU-run ρ, the set of violations along ρ is the set $\mathsf{V}(\rho) = \{i \in [0;|\rho|] \mid \tilde{\rho}_i > S\}$ of positions along ρ where the energy level exceeds the soft upper bound S. There are many ways to quantify violations along a run. As mentioned before, we consider three of them in this chapter, namely the total number of violations, the maximal number of consecutive violations, and the sum of the violations. We thus define the following three quantities: $\#\mathsf{V}(\rho) = |\mathsf{V}(\rho)|$, $\bar{\#}\mathsf{V}(\rho) = \max\{i-j \mid \forall k \in [i,j].\ k \in \mathsf{V}(\rho)\}$, and $\Sigma\mathsf{V}(\rho) = \sum_{i \in \mathsf{V}(\rho)}(\tilde{\rho}_i - U)$.
Figure 6.1 shows the evolution of $\#\mathsf{V}$ along a winning run in an $\mathsf{APNA}^{\#}$-energy game. One can notice that with a strong upper bound of 3, state q_t would not be reachable. On the other hand, if the strong upper bound is set to 6, then then there exists a run from q_0 to q_T, but that requires 3 violations of soft upper bound $S = 3$ (and the total amount of violations is 6).

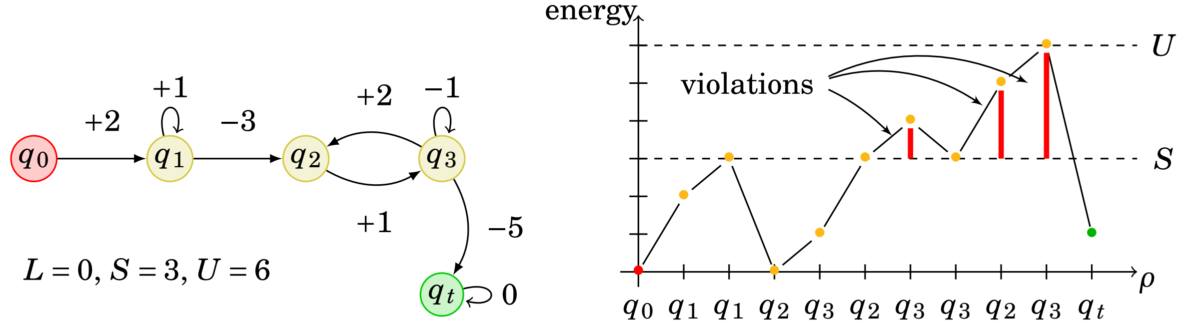

Figure 6.1: Energy level and $\#\mathsf{V}$ along a winning run in a $\mathsf{APNA}^{\#}$-energy reachability game.

Given three values $L \le S \le U$, the $\mathsf{APNA}^{\#}$-energy (resp. $\mathsf{APNA}^{\bar{\#}}$-energy, APNA^{Σ}-energy) objective

is the set of LU-feasible infinite paths π such that, along their associated runs ρ from (q_{init}, L), the number $\#\mathsf{V}(\rho)$ of violations (resp. maximal number of consecutive violations $\bar{\#}\mathsf{V}(\rho)$, sum $\Sigma\mathsf{V}(\rho)$ of violations) of the soft upper bound S is at most V.
Similarly, for a set of states R, the APNA$^{\#}$-energy (resp. APNA$^{\bar{\#}}$-energy, APNA$^{\Sigma}$-energy) reachability objective is the set of LU-feasible paths π reaching R such that along their associated run from (q_{init}, L), the number $\#\mathsf{V}(\rho)$ of violations (resp. maximal number of consecutive violations $\bar{\#}\mathsf{V}(\rho)$, sum $\Sigma\mathsf{V}(\rho)$ of violations) of the upper bound U is at most V.
We study the complexity of deciding the existence of a winning strategy for the objectives defined above, in both the one- and two-player settings. Further, for APNA*-energy games, we also address the following problems:

- **bound existence:** Given L, S and V, decide if there exists a value $U \in \mathbb{Z}$ such that Player 1 wins the APNA*-energy game;
- **minimization:** Given L and S, and a bound $V_{\max}$,

  find a value $U \in \mathbb{Z}$ such that Player 1 wins the game with the least possible violations less than $V_{\max}$, if any.

6.2 Decision Problems and Complexity

We now consider games with limited violations, i.e. (reachability) games with APNA$^{\#}$-energy, APNA$^{\bar{\#}}$-energy and APNA$^{\Sigma}$-energy objectives. We address the problems of deciding the winner in the one-player and two-player settings, and consider the existence and minimizations questions.

Theorem 6.2.1. APNA$^{\#}$-energy, APNA$^{\bar{\#}}$-energy and APNA$^{\Sigma}$-energy (reachability) games are PSPACE-complete for one-player arenas, and EXPTIME-complete for two-player arenas.

Proof. Membership in PSPACE and EXPTIME can be obtained by building a variant G_{APNA} of the G_{LU} arena: besides storing the energy level in each state, we can also store the amount of violations (for any of the three measures we consider). More precisely, given an arena G, lower and upper bounds L and U on the energy level, a soft bound S, and a bound V on the measure of violations, for any of our three measures of violations, the maximal energy level that can be reached along a path with violations smaller than or equal to V is $S + V \cdot w_{\max}$, where $w_{\max}$ is the maximal weight in our arena. We then define a new arena[1] G_{APNA} with set of states $(Q \times ([L;U] \cup \{\bot\}) \times ([0;V] \cup \{\bot\})^3)$, and each transition (q, w, q') of the original arena generates a transition from state $(q, l, (n, c, s))$ to state $(q', l', (n', c', s'))$ whenever

- l' correctly encodes the evolution of the energy level:
  - $l' = l + w$ if l and $l + w$ are in $[L;U]$;
  - $l' = \bot$ if either $l = \bot$ or $l + w < L$ or $l + w > U$;
- n' correctly stores the number of violations:
  - $n' = \bot$ if $l' = \bot$ or $n = \bot$;
  - $n' = n$ if $l' \in [L;S]$;
  - $n' = n + 1$ if $l' \in (S;U]$ and $n + 1 \leq V$;

[1]In order to factor our proof, we store all three measures of violations in one single arena; we actually have four variables because two of them are used to compute and store the maximal number of consecutive violations (which for solving the decision problem is not needed).

- $n' = \perp$ if $l' \in (S;U]$ and $n+1 > V$.

- c' is updated to count the current number of consecutive violations:
  - $c' = \perp$ if $l' = \perp$ or $c = \perp$;
  - $c' = 0$ if $l' \in [L;S]$;
  - $c' = c+1$ if $l' \in (S;U]$ and $c+1 \leq V$;
  - $c' = \perp$ if $l' \in (S;U]$ and $c+1 > V$.
- s' encodes the sum of all violations:
  - $s' = \perp$ if $l' = \perp$ or $s = \perp$;
  - $s' = s$ if $l' \in [L;S]$;
  - $s' = s + (l' - U)$ if $l' \in (S;U]$ and $s + (l' - S) \leq V$;
  - $s' = \perp$ if $l' \in (S;U]$ and $s + (l' - S) > V$.

In this arena, n, c and s keep tack of the number of violations, number of consecutive violations and sum of violations; their values are set fo $\perp$ as soon as they exceed the bound, or if the energy level has exceeded its bounds $[L'U]$. The arena G_{APNA} has size exponential, and our APNA*-energy-reachability problems can be reduced to solving reachability of the relevant set of states in that arena (e.g., Player 1 wins the APNA$^\#$-energy reachability game if, and only if, she wins in the modified game G_{APNA} for the objective of reaching the target set without visiting states where $n = \perp$).

Hardness results are obtained by setting the number/amount of allowed violations to zero, thereby recovering the classical LU-energy reachability games, which we proved are PSPACE-complete and EXPTIME-complete for one-player and two-player arenas, respectively.
Solving APNA$^\#$-energy, APNA$^{\bar{\#}}$-energy, APNA$^\Sigma$-energy games (without reachability objective) can be performed with arena G_{APNA} built above. Now, the objective in APNA$^\#$-energy, APNA$^{\bar{\#}}$-energy, APNA$^\Sigma$-energy games is to enforce infinite runs, that avoid states with $l = \perp$ and with $n = \perp, c = \perp, s = \perp$, depending on the chosen criterion on violation. Again, these strategies can be found in PSPACE for the one-player case, and in EXPTIME in the two-player case. For the hardness part, reduction from LU-energy still works. □

Given an arena G, a lower bound L, a soft upper bound S, and a maximal number of violations V (resp. maximal number of consecutive violations, maximal sum of overloads), we know that the energy level cannot exceed $U_{max} = S + V \cdot w_{max}$. But, clearly this is an overestimation. Hence, the natural questions that arise now are *Bound existence* and *Minimization* question, defined in the introduction of the chapter. Let us now see what is the complexity of deciding these two questions in the next section.

6.3 Bound Existence and Minimization

When the strong upper bound U is not given, the existence problem consists in deciding if such a bound exist under which Player 1 wins the APNA*-energy game. We have:

Theorem 6.3.1. The existence problems for APNA$^\#$-energy, APNA$^{\bar{\#}}$-energy, and APNA$^\Sigma$-energy (reachability) games are PSPACE complete for the one-player case and EXPTIME-Complete for the two-player case.

Proof. Given an arena G, a lower bound L, a soft upper bound S, and a maximal number of violations V (resp. maximal number of consecutive violations, maximal sum of overloads), we know that the energy level cannot exceed U_{max} . Let us build the expanded arena G_{APNA} of Theorem 6.2.1. If there is a strategy in a APNA$^{\#}$-energy reachability game, it will visit only states of G with an energy level smaller than U_{max} and with a number of violations that cannot exceed V. Hence, in G_{LU}, this corresponds to a path visiting states of the form $(q,l,(n,c,s))$ in which $l \leq U_{\max}$, and $n \leq V$. Clearly, one can find a path from $(q_0,L,(0,0,0))$ to states of the form $(q_T,l,(n,c,s))$ in PSPACE. Similar reasoning holds for APNA$^{\bar{\#}}$-energy,APNA$^{\Sigma}$-energy reachability games, considering the c and s component of states in G_{APNA}.
For the two-player case, an attractor for T in G_{APNA} can be computed in polynomial time (but on an arena of exponential size w.r.t. H_max and V). If state $(q_0,L,(0,0,0))$ appears in the attractor, then there exists a strategy to reach q_T without exceeding U_{max}, and with a number of violations n is smaller than V. So existence for APNA$^{\#}$-energy in the two players setting is solvable in EXPTIME. Notice that the maximal energy level reached when using a strategy need not be H_{max}. As in the one-player case similar reasoning holds for APNA$^{\bar{\#}}$-energy,APNA$^{\Sigma}$-energy reachability games, considering the c and s component of states in G_{APNA}.

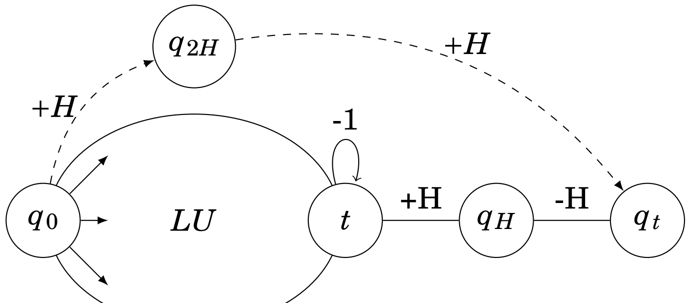

Figure 6.2: Reduction from an LU-energy reachability game with initial state q_0 and target state t to an H existence problem with initial state q_0 and target state q_t, and from an LU-energy reachability game to a minimization game (with additional dashed moves).

For the hardness part of the existence problem, we can transform a LU-energy reachability game with strong bounds L,U into an existence problem for an APNA$^{\#}$-energy reachability game with upper bound H and number of violations V. Let $G = (Q,E,w)$ be an arena, t be a target state, L,U be rational strong lower and upper bounds. As shown in Figure 6.2, we can choose a value $H > U$ and compute an arena $G' = (Q',E',w')$ where $Q' = Q \uplus \{q_h,q_t\}$, $E' = E \uplus \{(t,q_H),(t,t),(q_H,q_t)\}$ and $w'(q,q') = w(q,q')$ if $(q,q') \in E$, $w'(t,t) = -1$, $w'(t,q_H) = H-L$ and $w'(q_H,q_t) = -H$. Then we set $V = 1$, and choose q_t as target state for an APNA$^{\#}$-energy reachability game. Every winning strategy for Player 1 needs to use transition from t to q_H, which causes a violation. Hence every winning strategy for Player 1 needs to enforce a run from q_0 with initial energy budget B_0 to state t without causing any violation of upper bound U. So, there exists H allowing to find a strategy to win the APNA$^{\#}$-energy reachability game iff Player 1 has a strategy for the initial LU game. The reduction works both in the one player and two-player setting, and for APNA$^{\bar{\#}}$-energy,APNA$^{\Sigma}$-energy reachability games as well.

Let us now address the existence question for APNA$^{\#}$-energy (resp. APNA$^{\bar{\#}}$-energy,APNA$^{\Sigma}$-energy) games without reachability objective. Player 1 wins these games iff there exists an infinite path in which the number of violations (resp. the consecutive number of violations, the sum of violations) never exceeds V. As already before, the energy level in these runs cannot exceed H_{max}. So, cycles can be found in G_{APNA} in PSPACE for the one player case and in EXPTIME for the two-player case.

For the hardness part, one can easily transform a LU-energy game into an existence question by choosing an arbitrary value H, and then building a graph G_{red} by adding a pair of nodes q_i, q_H to the arena, edges $(q_i, q_H), (q_H, q_0)$ with respective weights $+H, -H$, and imposing a number of violations $V = 1$ (see Figure 6.3). Then, a maximal energy level greater than H allows for a strategy in the 1 player APNA$^{\#}$-energy game iff the LU-energy game on G has a solution, both for the one-player and two-players cases. The reduction works similarly for APNA$^{\overline{\#}}$-energy,APNA$^{\Sigma}$-energy games.

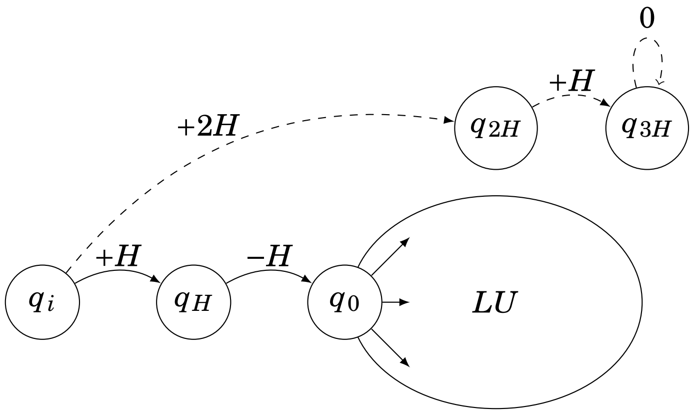

Figure 6.3: Reduction from an LU-energy game to an existence question in $APNA_{L,U}^{+H,X\leq V}$ (vithout dashed edges). Reduction from an LU-energy game to a minimization question in $APNA_{L,U}^{+H,X\leq V}$ (vith dashed edges).

□

Now, we move to the minimization question. We have:

Theorem 6.3.2. Let G be an arena, L and S be integer bounds, and $V_{\max}$ be an integer. There exist algorithms that compute the value of U that minimizes the value of V for which Player 1 has a winning strategy in a APNA*-energy (reachability) game. These algorithm runs in PSPACE for 1-player games and in EXPTIME for two-player games. These bounds are sharp.

Proof. Let us first consider APNA$^{\#}$-energy reachability games. Given H, V, one can check in PSPACE for the single player version whether a solution exists without exceeding energy level H nor bound V, and in EXPTIME fro the two-payer version. This can be done by computing an arena G_{APNA}^{H} that has maximal energy level $\min(H_{max}, H)$ (where $H_{max} = U + V_{max} \cdot w_{max}$).
To find the value for H that minimizes V, we can first notice that, for $H' > H$, the minimal possible number of violations is smaller in $G_{APNA}^{H'}$ than in G_{APNA}^{H}, as increasing the value of energy threshold can only add new runs in the arena. One can then perform a binary search for an optimal value in $[U, H_{max}]$. Given a tested value K, one can associate to each state (q, E, v) of player 1 in the attractor of T a value v if $E \leq K$ or ∞ otherwise. Similarly, one can associates value v to each state (q, E, v) of Player 2 where $E \leq K$ and which successors all have an energy level smaller than K, and ∞ otherwise. Then it suffices to find the path from $(q_0, B_0, 0)$ to T in the attractor which maximal violation level is minimal. This can be done with a slight adaptation of Dijkstra's algorithm [Dij59], where the shortest distance only maintains the maximal violation value encountered from $(q_0, B_0, 0)$ to the current state. This is done in polynomial time in the size of the attractor (which can be as large as arena $G_{\mathsf{APNA}^{\#}-energy, H_{max}}$).
If the value found is ∞ for a search with energy bound K in the attractor, then there is no way to win APNA$^{\#}$-energy with maximal number of overload K. Otherwise, one gets the optimal

number of violations with this bound K. One can perform a binary search in $[U, H_{max}]$, by testing successive values for K. The number of violations in arena $G_{\mathsf{APNA}^{\#}-energy,K}$ decreases as K increases. If there is no solution for $\mathsf{APNA}^{\#}$-energy with bound K then the optimal value is higher than K. At every step, we hence search an optimal value in an interval $[K_{min}, K_{max}]$, which size id divided by 2 at each iteration of the binary search. We can return value $H = K_{min}$ as soon as the optimal number of violations is the same in $G_{\mathsf{APNA}^{\#}-energy,K_{min}}$ and $G_{\mathsf{APNA}^{\#}-energy,K_{max}}$. i.e. after a logarithmic (in $H_{max} - U$) number of steps. One can hence find optimal values for V and H in an arena in polynomial space for the one player version of the game and in EXPTIME for the two-players game.

Let us now show that these bounds are sharp. We add to the arena in the proof of thm. 6.3.1 (Figure 6.2) above a state q_{2H} and a pair of transitions $(q_0, q_{2H}), (q_{2H}, q_t)$ with weights $w'(q_0, q_{2H}) = H$ and $w'(q_{2H}, q_t) = H$. With these new moves, the minimal number of violations is 1, with maximal energy $K = H$ if Player 1 wins the LU-energy game, and Player 2 with maximal energy $K = 2H$ otherwise. Hence, finding the minimal values for V, H is as hard as solving an LU-energy reachability game. The process to find a bound in $\mathsf{APNA}^{\bar{\#}}$-energy, APNA^{Σ}-energy reachability games is identical.
The minimization problems in the 1 player setting for $\mathsf{APNA}^{\#}$-energy, $\mathsf{APNA}^{\bar{\#}}$-energy, and APNA^{Σ}-energy games follow the same lines. Given V and H, one can check existence of a strategy for Player one in an $\mathsf{APNA} - energy^{X}$ game (for $X \in \{nbV, \bar{\#}\mathsf{V}, \Sigma\mathsf{V}\}$) in PSPACE for the one player setting, and in EXPTIME for the two-payer setting. If Player 1 has no winning strategy with bound H_{max} then he has no winning strategy for larger values. As for reachability games, we can show that the number of violations decreases when H increases. One can hence perform up to $\log(H_{max} - U)$ test to find the minimal value H such that player 1 has a winning strategy for some bound H and none for bound $H - 1$. Hence, the minimization question is also in EXPTIME. Let us now prove that this bound is tight. We reuse the same arena as the arena of Figure 6.3 in the proof of Theorem 6.3.1. We add two additional edges $(q_i, q_{2H})(q_{2H}, q_{0H})$ with respective weights $2H$ and $B_0 - 2H$ and a loop on q_{0H} of weight 0 to arena G_{red} above. Then the answer to the minimization question, is H if the 1 player LU-energy game has a solution and $2.H$ otherwise. This shows that minimization is as hard as solving an LU-energy game.

□

6.4 Conclusion

In this chapter, we have seen that all kind of APNA games are PSPACE-complete for the 1-player version and EXPTIME-complete for the 2-player versions. Then, we asked two natural questions: bound existence and minimization, both of which also turned out to have same complexity as the game problems, both for 1 and 2-player cases.

Chapter 7

Conclusion

In this thesis, we have considered several variants of energy games. The first variant defines games with upper and lower bound constraints, combined with reachability or infinite runs objectives. The second variant proposed defines games with a strong lower bound and a relaxed upper bound that can be weak or temporarily exceeded, combined with reachability or infinite run objectives, and constraints on violations of upper bound.

In the one player case, complexities ranges from PTIME to PSPACE-Complete. and in the two-player case from NP ∩ coNP to EXPTIME-Complete. In general, the complexity is the same for a reachability and for an infinite run objective. Interestingly, for LW-energy games, the complexity of the single player case is PTIME, but reachability objectives require exponential memory (in the size of the weak upper bound) while strategies are memoryless for infinite run objectives.

Table 7.1 summarizes known results, and the results obtained in this paper (where $\mathsf{APNA}^\star$-energy gathers all three energy constraints with violations). We furthermore show that the minimization problem for $\mathsf{APNA}^\star$-energy (reachability) games require algorithms that run in PSPACE in the one-player case, and in EXPTIME in the two-players case.

	Reachability		Infinite runs	
	1 player	2 players	1 player	2 players
L-energy	PTIME (thm. 3.1.2)	NP∩ coNP (thm. 3.1.2)	PTIME [BFL+08]	in NP ∩ coNP [BFL+08]
LU-energy	PSPACE-c. (thm. 4.1.1)	EXPTIME-c. (thm. 4.2.1)	PSPACE-c. [BFL+08]	EXPTIME-c. [BFL+08]
LW-energy	PTIME (thm. 5.2.1)	coNP (thm. 5.3.3)	PTIME [BFL+08]	NP ∩ coNP [BFL+08]
APNA^*-games	PSPACE-c. (thm. 6.2.1)	EXPTIME-c. (thm. 6.2.1)	PSPACE-c. (thm. 6.2.1)	EXPTIME-c. (thm. 6.2.1)
Bound existence	PSPACE-c. (thm. 6.3.1)	EXPTIME-c. (thm. 6.3.1)	PSPACE-c. (thm. 6.3.1)	EXPTIME-c. (thm. 6.3.1)

Table 7.1: Summary of the results

A possible extension of this work is to consider energy games with mean payoff function and discounted total payoff for the energy level and for the violation constraints, and the associated minimization and existence problems.

Chapter 8

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Appendix A

Appendix Title

A.1 Proofs of Section 5.2

Lemma 5.2.2: Let π be a finite path in a one-player arena G. If $(q,u) \xrightarrow{\pi}_{LW} (q',u')$, then for any $v \geq u$, $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ for some $v' \geq u'$.

Proof. Write $\pi = (e_i)_{0 \leq i < n}$, with $e_i = (q_i, p_i, q'_i)$ for each i. The sequence defined as

$$u_0 = u \qquad u_{i+1} = \min W, u_i + p_i$$

is the sequence of energy levels along the run $(q,u) \xrightarrow{\pi}_{LW} (q',u')$. For $v \geq u$, letting

$$v_0 = v \qquad v_{i+1} = \min W, v_i + p_i,$$

we easily prove by induction that for all i, $u_i \leq v_i \leq W$, which entails that $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ with $v' = v_n \geq u_n = u'$. □

Proof of Lemma 5.2.3 Let π be a finite path in a one-player arena G, and consider two LW-runs $(q,u) \xrightarrow{\pi}_{LW} (q',u')$ and $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ with $u \leq v$. Then $u' - u \geq v' - v$, and if the inequality is strict, then the energy level along the run $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ must have hit W.

Proof. The first statement is proven by induction: we again write $\pi = (e_i)_{0 \leq i < n}$, with $e_i = (q_i, p_i, q'_i)$ for each i, and

$$u_0 = u \qquad u_{i+1} = \min(W, u_i + p_i)$$
$$v_0 = v \qquad v_{i+1} = \min(W, v_i + p_i).$$

Then $u_{i+1} - u_i = \min(W - u_i, p_i)$ and $v_{i+1} - v_i = \min(W - v_i, p_i)$. Since $u_i \leq v_i$ for all i, we also have $W - u_i \geq W - v_i$, and $u_{i+1} - u_i \geq v_{i+1} - v_i$. By summing up these inequalities, we get $u_{i+1} - u_0 \geq v_{i+1} - v_0$. Now, as long as $W - v_i \geq p_i$ (then also $W - u_i \geq p_i$), the inequalities above are equalities. It follows that if the inequality is strict, then the run $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ must have hit W. □

Proof of Lemma 5.2.4 Let π be a finite path in a one-player arena G, for which there is an LW-runs $(q,u) \xrightarrow{\pi}_{LW} (q',u')$. If u' is the maximal energy level along that run, then $(q,W) \xrightarrow{\pi}_{LW} (q',W)$; if u is the maximal energy level along the run above, then $(q,W) \xrightarrow{\pi}_{LW} (q', W + u' - u)$.

Proof. Write $\pi = (e_i)_{0 \leq i < n}$, with $e_i = (q_i, p_i, q'_i)$ for each i. If u' is the maximal energy level, then for all i, we have $\sum_{j=i}^{n-1} p_j \geq 0$. Now, define

$$v_0 = W \qquad v_{i+1} = \min(W, v_i + p_i).$$

If $v_n < W$, then by induction we also have $v_i < W$ for all i, contradicting the fact that $v_0 = W$. This proves our first result.
Similarly, if u is the maximal energy level, then for all i, we have $\sum_{j=0}^{i} p_j \leq 0$. Then for all i, $v_{i+1} = v_i + p_i \leq W$, so that $v_n - v_0 = u' - u$. Our second result follows. □

Proof of Lemma 5.2.5 Let π be a finite path in a one-player arena G. If $(q,u) \xrightarrow{\pi}_{LW} (q',u')$ with $u' > u$ and $(q,w) \xrightarrow{\pi}_{LW} (q',w')$ with $w' > w$, then for any $u \leq v \leq w$, it holds $(q,v) \xrightarrow{\pi}_{LW} (q',v')$ with $v' > v$.

Proof. Using Lemma 5.2.2, we immediately have $(q,v) \xrightarrow{\pi}_{LW} (q',v')$. As in the previous proof, we define sequences

$$\begin{aligned} u_0 &= u & u_{i+1} &= \min(W, u_i + p_i) \\ v_0 &= v & v_{i+1} &= \min(W, v_i + p_i) \\ w_0 &= w & w_{i+1} &= \min(W, w_i + p_i). \end{aligned}$$

We still have $u_i \leq v_i \leq w_i$ for all i. Moreover, if $v_j < W$ for all $j \leq i$, then $v_i - u_i = v - u$. As a consequence, if $v' \leq v$, then it must be the case that $v_j = W$ for some j; but then $w_j = v_j$, since $v_j \leq w_j \leq W$. It follows that $w_k = v_k$ for all $k \geq j$, so at the end of π we have $w' = v'$. Assuming $v' \leq v$ raises a contradiction since we have $v' = w' > w \geq v$. Hence $v' > v$. □

Proof of Lemma 5.2.6 Let π be a finite path in a one-player arena G. If $(q,u) \xrightarrow{\pi}_{LW} (q',u')$ and π can be decomposed as $\pi_1 \cdot \pi_2 \cdot \pi_3$ in such a way that $(q,u) \xrightarrow{\pi_1}_{LW} (s,v) \xrightarrow{\pi_2}_{LW} (s,v') \xrightarrow{\pi_3}_{LW} (q',u')$ with $v' \leq v$, then $(q,u) \xrightarrow{\pi_1 \cdot \pi_3}_{LW} (q',u'')$ with $u'' \geq u'$.

Proof. Since $(s,v') \xrightarrow{\pi_3}_{LW} (q',u')$ and $v' \leq v$, by Lemma 5.2.2 we also have $(s,v) \xrightarrow{\pi_3}_{LW} (q',u'')$ for some $u'' \geq u'$. The result follows. □

Proof of Lemma 5.2.7 Let π be a cycle on q such that $(q,u) \xrightarrow{\pi}_{LW} (q,v)$ for some $u \leq v$. Then $(q,u) \xrightarrow{\pi^{W-L}}_{LW} (q,v')$ for some v', and $(q,v') \xrightarrow{\pi}_{LW} (q,v')$.

Proof. The case where $u = v$ is trivial. We assume $u < v$. Applying Lemma 5.2.2 inductively, we get that the cycle can be iterated arbitrarily many times; this also proves that the sequence of energy levels reached at the end of each iteration is non-decreasing.
Now, assume that $(q,v') \xrightarrow{\pi}_{LW} (q,v'')$ for some $v'' \neq v'$. Then $v'' > v'$. Lemma 5.2.5 then entails that the sequence of energy levels reached at the end of each iteration is increasing. Since the loop has been iterated $W - L$ times, the energy level in v'' would exceed W, which is impossible. This proves our result. □

Proof of Lemma 5.2.9 Let $[q,d]$ be a state of the DAG, and M and m be two integers such that $0 \leq m \leq M$. Upon termination of this algorithm, state $[q,d]$ of the DAG is labelled with (M,m) if, and only if, there is an LW-run of length d from (q_0,L) to $(q,M-m)$ along which the energy level always remains in the interval $[L,M]$.

Proof. The proof is by induction on d. The result is trivial for $d = 0$. Now, assume it holds for some depth $d-1$, and pick a state $[q,d]$. For the first direction, if $[q,d]$ is labelled with (M,m), then this label was added using some transition $([q',d-1],w,[q,d])$ and some label (M',m') of $[q',d-1]$. By induction, there is an LW-run ρ of length $d-1$ from (q_0,L) to $(q',M'-m')$ in G along which the energy level remains in the interval $[L,M']$. We consider two cases, corresponding to the two ways of updating the pair of values:

- if $w > m'$, then we have $M = min(W, M' - m' + w)$ and $m = 0$. Now, the transition $([q', d-1], w, [q, d])$ in the DAG originates from a transition (q', w, q) in G; taking this transition after ρ provides us with the run of length d from (q_0, L) to $(q, M-m)$ along which the energy level remains in $[L, M]$, as required;

- if $m' + L - M' \le w \le m'$, then $M = M'$ and $m = m' - w$. Again, taking transition (q', w, q) after ρ provides us with the LW-run we are looking for.

Conversely, if there is an LW-run ρ of length d from (q_0, L) to $(q, M-m)$ along which the energy level always remains in the interval $[L, M]$, then we write $\rho = \rho' \cdot ((q', l'), w, (q, M-m))$, distinguishing its last transition. By induction, $[q', d-1]$ must have been labelled with a pair (M', m') such that $l' = M' - m'$ and the energy level along ρ' remained within $[L, M']$. Now, from the existence of a transition $((q', l'), w, (q, M-m))$, we know that there is a transition $([q', d-1], w, [q, d])$ in the DAG, which will generate the required label of $[q, d]$. □

Proof of Lemma 5.2.10 Let $[q_0, d]$ be a state of the DAG, with $d > 0$. Let m be a non-negative integer such that $L < W - m$. Upon termination of this algorithm, state $[q_0, d]$ is labelled with (M, m) such that $M - m > L$ if, and only if, there is a universal positive cycle ϕ on q_0 of length d such that $(q_0, L) \xrightarrow{\phi^{W-L}}_{LW} (q_0, W-m)$.

Proof. First assume that $[q_0, d]$ is labelled with (M, m) for some M such that $M - m > L$. From Lemma 5.2.9, there is a cycle ϕ on q_0 of length d generating a run $(q_0, L) \xrightarrow{\phi}_{LW} (q_0, M-m)$ along which the energy level is within $[L, M]$. Then $M - m \ge L$, so that Lemma 5.2.7 applies: we then get $(q_0, L) \xrightarrow{\phi^{W-L}}_{LW} (q_0, E)$ with $(q_0, E) \xrightarrow{\phi}_{LW} (q_0, E)$. Write $(p_i)_{0 \le i < |\phi|}$ for the sequence of weights along ϕ. Also write ρ for the run $(q_0, L) \xrightarrow{\phi}_{LW} (q_0, M-m)$, and σ for the run $(q_0, E) \xrightarrow{\phi}_{LW} (q_0, E)$. As $L < M - m$, then by Lemma 5.2.3, it must be the case that energy level W is reached along σ. Write i_0 and j_0 for the first and last positions along ρ for which the energy level along ρ is M. That is, the subpath from index 0 to index i_0 has growing energy level, and the subpath from j_0 to $|\rho|$ has a decreasing energy level. Assume $\tilde{\sigma}_{i_0} \ne W$: by Lemma 5.2.2, we must have $M = \tilde{\rho}_{i_0} \le \tilde{\sigma}_{i_0} < W$. Then for all $k \ge i_0$, $\sum_{l=i_0}^{k} p_l \le 0$, and $\sum_{l=i_0}^{j_0} p_l = 0$. Since $\tilde{\sigma}_{i_0} < W$, then also $\tilde{\sigma}_k < W$ for all $k \ge i_0$. According to Lemma 5.2.3, energy level W is reached in σ, so there exists some $k_0 < i_0$ such that $\tilde{\sigma}_{k_0} = W$; However, as i_0 is the index of the first maximal value in ρ, we have $\tilde{\rho}_{k_0} < M$, and the energy level increases in run ρ between k_0 and i_0. So according to Lemma 5.2.4, we should have $\tilde{\sigma}_{i_0} < W$, which raises a contradiction. Hence we proved $\tilde{\sigma}_{i_0} = W$; applying the second result of Lemma 5.2.4, we get $E = W - m$.

Conversely, if there is a universal positive cycle ϕ satisfying the conditions of the lemma, let F be such that $(q_0, L) \xrightarrow{\phi}_{LW} (q_0, F)$, and M be the the maximal energy level encountered along the run $(q_0, L) \xrightarrow{\phi}_{LW} (q_0, F)$. By Lemma 5.2.9, state $[q_0, d]$ is labelled with (M, m') for some $m' \ge 0$ such that $F = M - m'$. By Lemma 5.2.7, we must have $(q_0, L) \xrightarrow{\phi^{W-L}}_{LW} (q_0, W-m')$. □

Proof of Lemma 5.2.11 Consider two paths π and π' such that $first(\pi) = first(\pi')$ and $last(\pi) = last(\pi')$, and with respective values (M, m) and (M', m') such that $(M, m) \preceq (M', m')$. If π is a prefix of a universal cycle ϕ, then π' is a prefix of a universal cycle ϕ' with $\phi' \rhd \phi$.

Proof. Let $q = first(\pi)$ and $q' = last(\pi)$. We write ψ for the path such that $\phi = \pi \cdot \psi$; ψ is a path from q' to q. Then $(q, L) \xrightarrow{\pi}_{LW} (q'', M-m) \xrightarrow{\psi}_{LW} (q', F)$. Also, $(q, L) \xrightarrow{\pi}_{LW} (q'', M'-m')$. Since $M - m \le M' - m'$, we have $(q'', M'-m') \xrightarrow{\psi}_{LW} (q', F')$. We can thus let $\phi' = \pi' \cdot \psi$: by Lemma 5.2.10, the final energy level reached after iterating ϕ' is higher than the energy level reached after iterating ϕ, since $m' \le m$. Hence $\phi' \rhd \phi$. □

Proof of Lemma 5.2.12 If the DAG construction algorithm only stores maximal labels (for $\preceq$), then it runs in polynomial time.

Proof. We prove that, when attaching to each node $[q,d]$ of the DAG only the maximal labels (w.r.t $\preceq$) reached for a path of length d ending in state q, the number of values for the first component of the different labels that appear at depth $d > 0$ in the DAG is at most $d \cdot |Q|$. Since it only stores optimal labels, our algorithm will never associate to a state $[q,d]$ two labels having the same value on their first component. So, any state at depth d will have at most $d \cdot |Q|$ labels. So we prove, by induction on d, that the number of different values for the first component among the labels appearing at depth $d > 0$ is at most $d \cdot |Q|$. This is true for $d = 1$ since the initial state $(q,0)$ only contains $(M = 0, m = 0)$, and each transition with nonnegative weight w will create one new label $(w,0)$ (transitions with negative weight are not prefixes of universal cycles). Now, since all those labels have value 0 as second component, each state $[q,1]$ in the DAG will be attached at most one label. Hence, the total number of labels (and the total number of different values for their first component) is at most $|Q|$ at depth 1 in the DAG.
Now, assume that the labels appearing at depth $d > 1$ are all drawn from a set of labels $L = \{M_i, m_i) \mid 1 \le i \le n\}$ in which the number of different values of M_i is at most $d \cdot |Q|$. Consider a state $[q', d]$, labelled with $\{(M_i, m_i) \mid 1 \le i \le n_{q',d}\}$ (even if it means reindexing the labels). Pick a transition from $[q', d]$ to $[q'', d+1]$, with weight w. For each pair (M_i, m_i) associated with $[q,d]$, it creates a new label in $(q'', d+1)$; this label is

- either $(M_i - m_i + w, 0)$ if $m_i < w$;
- or $(M_i, m_i + w)$ if $m_i = M_i \le w \le m_i$.

Now, for a state $(q'', d+1)$, the set of labels created by all incoming transitions can be grouped as follows:

- labels having zero as their second component; among those, our algorithm only stores the one with maximal first component, as $(M_i, 0) \preceq (M_j, 0)$ as soon as $M_i \le M_j$;
- for each M_i appearing at depth d, labels having M_i as their first component; again, we only keep the one with minimal second component, as $(M, m_i) \preceq (M, m_j)$ when $m_j \le m_i$.

In the end, for this state $[q'', d+1]$, we keep at most one label for each distinct value among the first components M_i of labels appearing at depth d, and possibly one extra label with second value 0. In other terms, at depth $d+1$ the values that appear as first component of labels are obtained from values at depth d, plus possibly one per state; Hence, at depth $d+1$, there exists at most $(d+1) \cdot |Q|$ labels, which completes the proof of the induction step. □

Proof of Lemma 5.3.1 Let G be a two-player arena, equiped with an LW-energy-reachability objective. Let q be a state of G, and $u \le u'$ in $[L;W]$. If Player 1 wins the game from (q,u), then she also wins from (q,u').

Proof. Let σ be a winning strategy for Player 1 from (q,u). If she plays the same strategy from (q,u'), then for any strategy of Player 2, the resulting outcome from (q,u') follows the same transitions as the outcome of the same strategies from u, with higher energy level. Since σ is winning from (q,u), it is also winning from (q,u'). □